CHAPTER 26 : DIRECT-CURRENT CIRCUITS

- Direct current (dc) → current direction does not change with time

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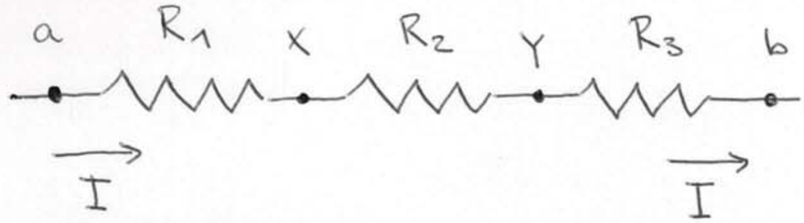
- Alernating current (ac) → current direction oscillates in time

26.1 Resistors in Series and Parallel

- Equivalent resistance R_{eq}
  - a single resistor that could replace a network of resistors so that the total current drawn by it and potential difference across its terminals are identical to those of the network

$$\boxed{V_{ab} = IR_{eq} \rightarrow R_{eq} = \frac{V_{ab}}{I}}$$

a) Resistors in series



Only a single current path exists through the network

→ the current through all the resistors is identical

→ $V_{ax} = IR_1$ $\quad V_{xy} = IR_2$ $\quad V_{yb} = IR_3$

The total potential difference

$$V_{ab} = V_{ax} + V_{xy} + V_{yb} = I\,(R_1 + R_2 + R_3)$$

$$\Downarrow$$

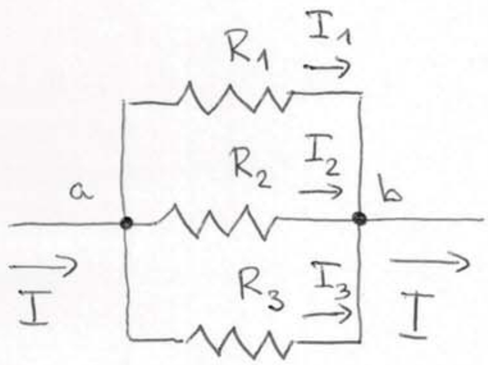
$$\boxed{R_{eq} = \frac{V_{ab}}{I} = R_1 + R_2 + R_3} \quad \text{In general} \quad \boxed{R_{eq} = \sum_{i=1}^{N} R_i}$$

The equivalent resistance of any number of resistors in series equals the sum of their individual resistances

$$R_{eq} > \max(R_i)$$

→ different from capacitors in series !!!

b) Resistors in Parallel



The current through each resistor is generally different

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The potential difference across each resistor is the same and equal to V_{ab}

$$\rightarrow \quad I_1 = \frac{V_{ab}}{R_1} \qquad I_2 = \frac{V_{ab}}{R_2} \qquad I_3 = \frac{V_{ab}}{R_3}$$

Charge conservation : current flowing into point $\underline{\underline{a}}$ is equal to current flowing out of point $\underline{\underline{a}}$

$$\downarrow$$

$$I = I_1 + I_2 + I_3 = V_{ab}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\boxed{\frac{1}{R_{eq}} = \frac{I}{V_{ab}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

In general $\boxed{\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i}}$

For any number of resistors in parallel, the reciprocal of the equivalent resistance equals the sum of the reciprocals of their individual resistances.

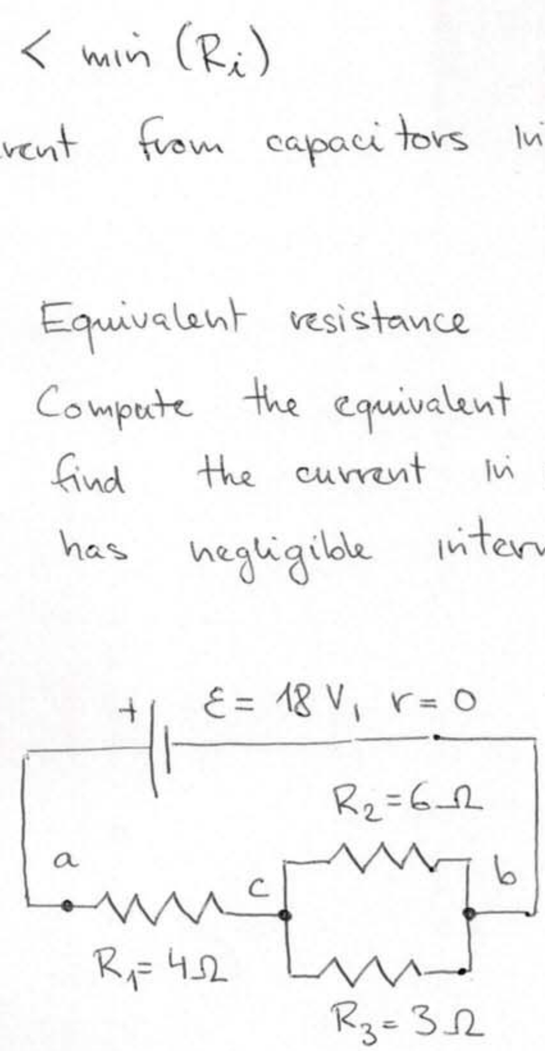
$R_{eq} < \min(R_i)$

→ different from capacitors in parallel !!!

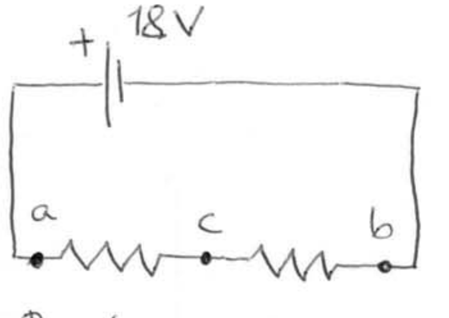
Ex 26.1 Equivalent resistance

Compute the equivalent resistance of the network and find the current in each resistor. The source of emf has negligible internal resistance.

a)



→ b)

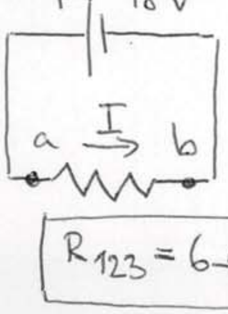


$R_1 = 4\,\Omega \qquad R_{23} = 2\,\Omega$

$$\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3} \Rightarrow R_{23} = \frac{R_2 R_3}{R_2 + R_3}$$

c)

→

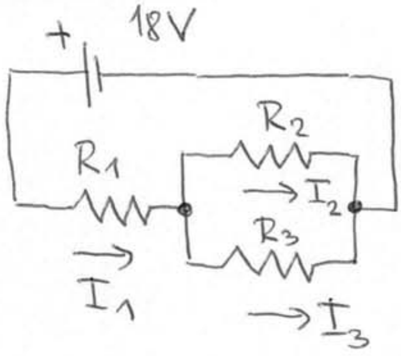


$\boxed{R_{123} = 6\,\Omega}$

$R_{123} = R_1 + R_{23}$

$$V_{ab} = \mathcal{E} \Rightarrow \boxed{I = \frac{V_{ab}}{R_{123}} = \frac{\mathcal{E}}{R_{123}} = 3\text{A}}$$

d)



$$\boxed{I_1 = I = 3A}$$

$$I_2 + I_3 = I \wedge R_2 I_2 = R_3 I_3$$

$$\boxed{I_2 = \frac{R_3 I}{R_2 + R_3} = 1A}$$

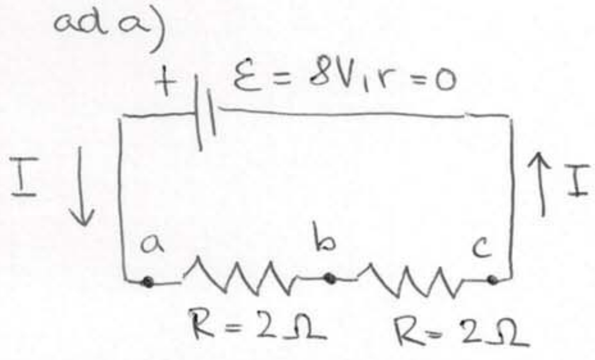
$$\boxed{I_3 = \frac{R_2 I}{R_2 + R_3} = 2A}$$

Ex 26.2: Two identical light bulbs are connected to a source with $\mathcal{E} = 8V$ and negligible internal resistance. Each bulb has a resistance $R = 2\,\Omega$.

Find the current through each bulb, the potential difference across each bulb, and the power delivered to each bulb and to the entire network if the bulbs are connected

a) in series or b) in parallel. What happens if one of the bulbs burns out?

ad a)



$$R_{eq} = 2R = \underline{\underline{4\,\Omega}}$$

- Current is identical for both bulbs: $I = \dfrac{V_{ac}}{R_{eq}} = \dfrac{\mathcal{E}}{R_{eq}} = \underline{\underline{2A}}$

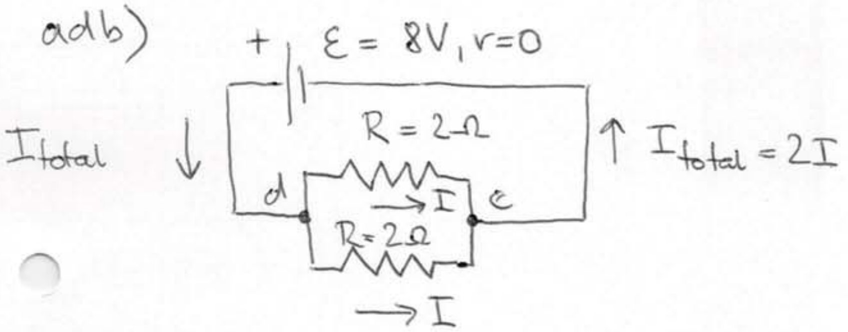
- Potential drops are also identical for both bulbs as they have the same resistance: $V_{ab} = V_{bc} = RI = \underline{\underline{4V}}$

- Power delivered to each bulb:

$$P = I^2 R = \frac{V_{ab}^2}{R} = \frac{V_{bc}^2}{R} = \underline{\underline{8W}} \Rightarrow \text{total power } P_{total} = 2 \cdot P = \underline{\underline{16W}}$$

- If one of the bulbs burns out, the other bulb has no current flowing through it

ad b)



$$\frac{1}{R_{eq}} = \frac{2}{R} \Rightarrow R_{eq} = \frac{R}{2} = \underline{\underline{1\Omega}}$$

- Total current $I_{total} = \frac{V_{de}}{R_{eq}} = \frac{\mathcal{E}}{R_{eq}} = 8A \Rightarrow I = \frac{I_{total}}{2} = \underline{\underline{4A}}$

- Potential drop is identical across both bulbs: $V_{de} = \mathcal{E} = \underline{\underline{8V}}$

- Power delivered to each bulb:

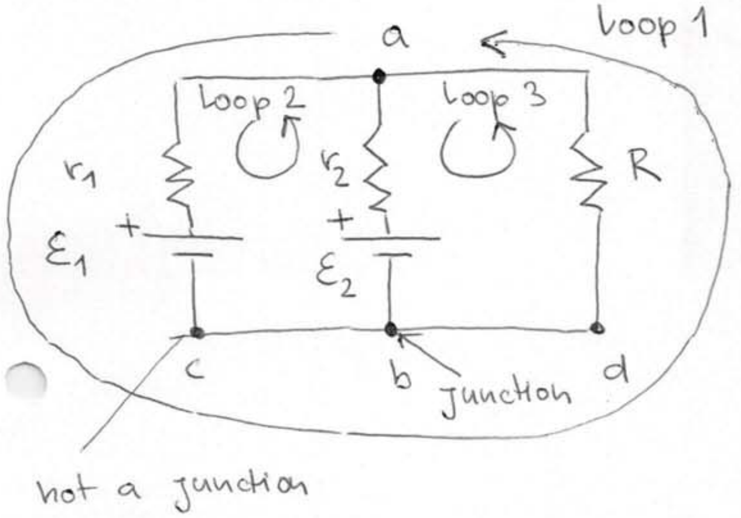
$$P = I^2 R = \frac{V_{de}^2}{R} = \underline{\underline{32W}} \Rightarrow \text{total power } P_{total} = 2P = \underline{\underline{64W}}$$

$\rightarrow$ power in each bulb is 4 times higher than in series combination

- If one of the bulbs burns out, the other bulb has current $I = \frac{\mathcal{E}}{R} = \underline{\underline{4A}}$ flowing through it and power $P = I^2 R = \underline{\underline{32W}}$ delivered to it $\rightarrow$ no change

26.2 Kirchhoff's Rules

- Rules for calculating currents and voltage drops in general electric circuits that cannot be reduced to series or parallel combinations



Loop = any closed conducting path

junction = point in a circuit where three or more conductors meet (also called nodes or branching points)

① Kirchhoff's junction rule

"The algebraic sum of the currents into any junction is zero"

$$\boxed{\sum I = 0}$$

algebraic sum: currents entering the junction are positive, currents leaving the junction are negative

$I_1 \longrightarrow \bullet \longrightarrow I_2$, $\downarrow I_3$

$I_1 - I_2 - I_3 = 0$

- based on the conservation of electric charge → no charge can accumulate at the junction → analogy with water flow through pipes (what comes in, must go out)

② Kirchhoff's loop rule

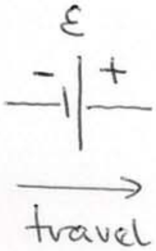
"The algebraic sum of the potential differences in any loop, including those associated with emfs and those of resistive elements, must equal zero"

$$\boxed{\sum V = 0}$$

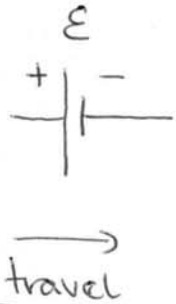
- follows from conservative character of the electrostatic force → when we go around a loop, measure potential differences, and return back to the starting point, total potential difference during the round trip is zero

Sign conventions for the loop rule

a) electromotive forces (emfs)

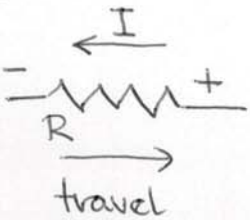


$+\mathcal{E}$ – travel direction from ⊖ to ⊕
↓
rise of potential

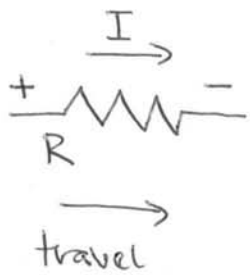


$-\mathcal{E}$ – travel direction from ⊕ to ⊖
↓
drop of potential

b) resistors



$+IR$ – travel opposite to current direction
↓
rise of potential



$-IR$ – travel in current direction
↓
drop of potential

Applying Kirchhoff's rules

① Label all assumed directions of currents and emfs
→ they don't have correspond to the actual directions
→ if the true direction is opposite to the assumed one, calculation result will come out with negative sign

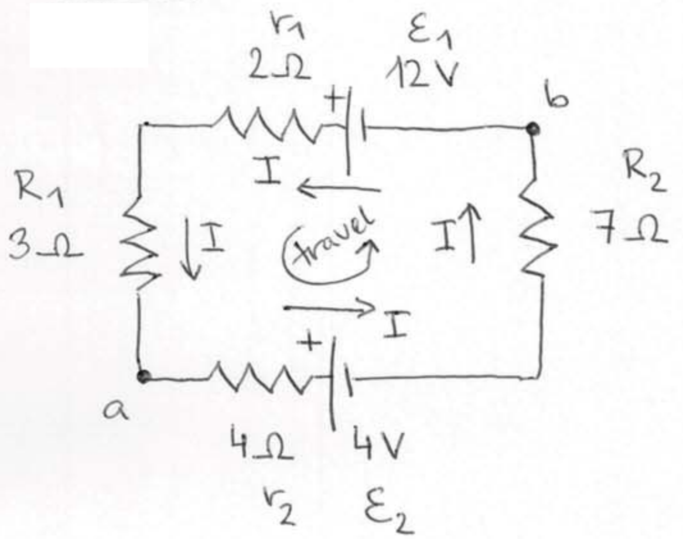
② Choose any closed loop in the network and designate a direction (clockwise ~~and~~ or counterclockwise) to travel around the loop

③ Add potential differences with correct sign as you travel across them (pay attention to sign convention)
→ the total sum on a round trip must be zero

(4) Apply the above strategy to as many different loops as necessary to have enough equations for calculating all unknown quantities

(5) Once all emfs and currents are calculated, they can be verified using a loop independent from those used in the previous calculations

Ex 26.3 : A single-loop circuit



a) Find the current in the circuit

$$-Ir_2 - \mathcal{E}_2 - IR_2 + \mathcal{E}_1 - Ir_1 - IR_1 = 0$$

starting from point (a)

$$-I(r_2 + R_2 + r_1 + R_1) = \mathcal{E}_2 - \mathcal{E}_1$$

$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{r_2 + R_2 + r_1 + R_1} = \frac{12V - 4V}{4\Omega + 7\Omega + 2\Omega + 3\Omega} = \underline{\underline{0{,}5\,A}}$$

- resulting current is positive → assumed current direction is correct

b) Find the potential difference V_{ab}

starting from point b : $V_b + \mathcal{E}_1 - r_1 I - R_1 I = V_a$

$$V_{ab} = V_a - V_b = \mathcal{E}_1 - I(r_1 + R_1) =$$

$$= 12\,V - 0{,}5\,A\,(2\,\Omega + 3\,\Omega)$$

Potential at a is 9,5V higher than the potential at b $\longrightarrow$ $= \underline{\underline{9{,}5\,V}}$

c) Find the power output of the emf of each battery

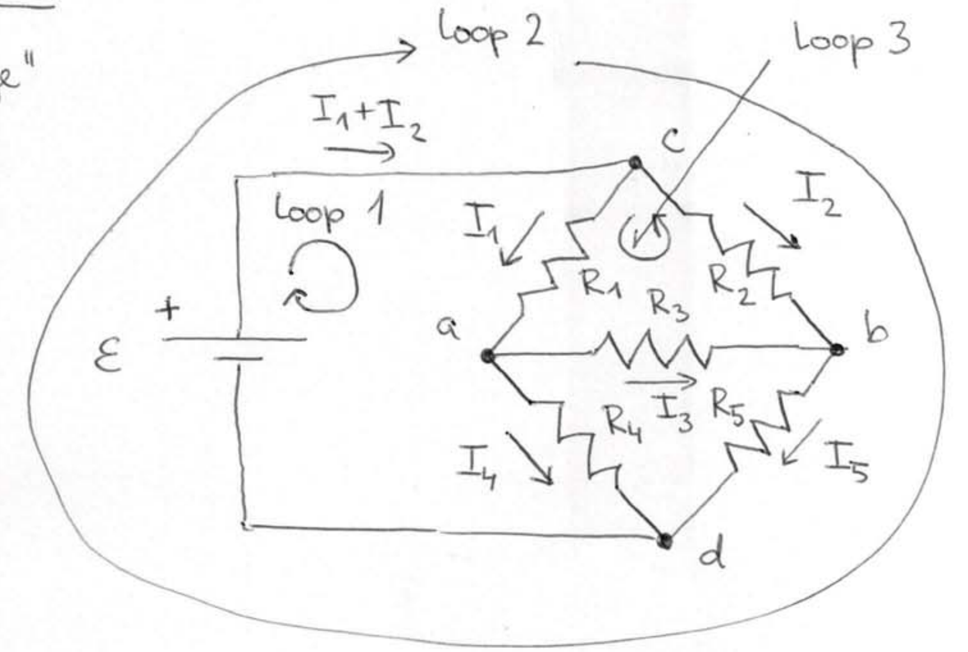
emf $\mathcal{E}_1$: $P = \mathcal{E}_1 I = 12\,V \cdot 0{,}5\,A = 6\,W$

emf $\mathcal{E}_2$: $P = -\mathcal{E}_2 I = -4\,V \cdot 0{,}5\,A = -2\,W$

current runs through the battery from higher to lower potential side $\rightarrow$ work is done on the battery $\rightarrow$ battery is being recharged

Ex 26.6 A complex network - find all currents and equiv. resistance

"Bridge" circuit



$R_1 = R_2 = R_3 = R_4 = 1\Omega$

$R_5 = 2\Omega$

$I_4 = I_1 - I_3$

$I_5 = I_2 + I_3$

$\mathcal{E} = 13\,V$

- Loop 1: $\mathcal{E} - I_1 R_1 - (I_1 - I_3) \cdot R_4 = 0$
- Loop 2: $\mathcal{E} - I_2 R_2 - (I_2 + I_3) R_5 = 0$
- Loop 3: $- I_1 R_1 - I_3 R_3 + I_2 R_2 = 0$

system of 3 algebraic equations for 3 unknown currents I_1, I_2, I_3

From (3): $I_2 = \dfrac{I_1 R_1 + I_3 R_3}{R_2}$

$$\Downarrow$$

(2): $\mathcal{E} - (I_1 R_1 + I_3 R_3) - \left(\dfrac{I_1 R_1 + I_3 R_3}{R_2} + I_3 \right) R_5 = 0$

(1): $\mathcal{E} - I_1 R_1 - I_1 R_4 + I_3 R_4 = 0$

$$\Downarrow$$

$$I_3 = \frac{I_1 (R_1 + R_4) - \mathcal{E}}{R_4}$$

$$\Downarrow$$

(2): $0 = \mathcal{E} - I_1 R_1 - \dfrac{R_3}{R_4} \left[I_1 (R_1 + R_4) - \mathcal{E} \right] - I_1 \dfrac{R_1 R_5}{R_2} - \left(\dfrac{R_3}{R_2} + 1 \right) \dfrac{R_5}{R_4} \left[I_1 (R_1 + R_4) - \mathcal{E} \right]$

$$0 = \cancel{13} - I_1 - [I_1 \cdot 2 - \cancel{13}] - I_1 2 - 4 \cdot [I_1 \cdot 2 - \cancel{13}]$$

$$0 = 78 - 13 I_1 \Rightarrow I_1 = \frac{78\,V}{13\,\Omega} = \underline{\underline{6\,A}}$$

$$I_3 = \frac{I_1 (R_1 + R_4) - \mathcal{E}}{R_4} = \frac{6\,A \cdot 2\,\Omega - 13\,V}{1\,\Omega} = \underline{\underline{-1\,A}}$$

$$I_2 = \frac{I_1 R_1 + I_3 R_3}{R_2} = \frac{6\,A \cdot 1\,\Omega + (-1\,A) \cdot 1\,\Omega}{1\,\Omega} = \underline{\underline{5\,A}}$$

The total current through the network $I_{total} = I_1 + I_2 = 11\,A$

$$\Downarrow$$

$$V_{cd} = R_{eq} \cdot I_{total} \rightarrow R_{eq} = \frac{V_{cd}}{I_{total}} = \frac{13\,V}{11\,A} = \underline{\underline{1{,}2\,\Omega}}$$

26.3. Electrical Measuring Instruments

- Often based on galvanometer → a pivoted coil of wire placed in the magnetic field of a permanent magnet

$\downarrow$

When current runs through the coil, magnetic field exerts a torque that tends to rotate the coil; this torque is proportional to the current

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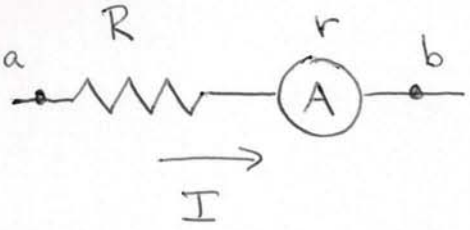
Magnetic torque is counter-ballanced by mechanical torque generated by a torsion spring; this torque is proportional to the angle of the coil rotation

$\Rightarrow$ greater current = greater magnetic torque
= greater coil rotation angle $\rightarrow$ <u>current measurement</u>

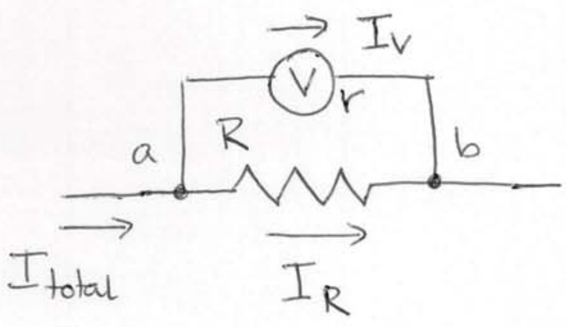
- If Ohm's Law holds, current through the coil is directly proportional to the potential difference between coil terminals
  $\longrightarrow$ <u>voltage measurement</u>

- Current measurement $\rightarrow$ <u>ammeters</u> $\rightarrow$ ideally zero (very low) resistance not to affect the current in the measurement branch in a circuit (ammeter is connected <u>in series</u>)

- Voltage measurement $\rightarrow$ voltmeters $\rightarrow$ ideally infinite (very high) resistance not to alter any of the currents in the circuit (voltmeter is connected <u>in parallel</u>)



$$I = \frac{V_{ab}}{R+r} \Rightarrow r \ll R \text{ required}$$
not to affect the current

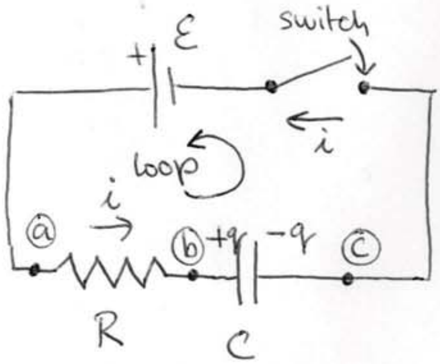


$$I_{total} = I_R + I_V = \frac{V_{ab}}{R} + \frac{V_{ab}}{r} =$$
$$= V_{ab}\left(\frac{1}{R} + \frac{1}{r}\right)$$
$$\Downarrow$$
$r \gg R$ required to achieve
$$I_{total} \approx I_R$$

26.4. R–C Circuits

- Circuits that involve charging or discharging of a capacitor through a resistor → currents, voltages, and powers change with time

Charging a capacitor



- Initially, the capacitor is uncharged
- At time $t=0$, the switch is closed → the capacitor is being charged through the resistor from the battery with zero internal res.

- Instantaneous time-varying current $\underline{\underline{i}}$ – identical value in every part of the circuit
- Identically, we have a time-varying charge and voltage $\underline{\underline{q}}$ and $\underline{\underline{v_{bc}}}$ on the capacitor
- At $t=0$, the capacitor is uncharged ($v_{bc}=0$)
  → Kirchhoff's loop law gives $v_{ab} = \mathcal{E} = RI_0$
  → initial current through the resistor $\boxed{I_0 = \mathcal{E}/R}$

- As the capacitor charges, its voltage $\underline{\underline{v_{bc}}}$ increases and the potential drop across the resistor $v_{ab} = R\cdot i$ decreases. At any given time, Kirchhoff's loop law gives

$$\mathcal{E} = v_{ab} + v_{bc} = i\cdot R + \frac{q}{C}$$

→ in the end of the charging process,

$$\mathcal{E} = \frac{q}{C} \Rightarrow \text{current } i \to 0 \text{ , final charge } \boxed{Q_f = \mathcal{E} \cdot C}$$

Kirchhoff's loop law:

$$\mathcal{E} - iR - \frac{q}{C} = 0 \quad \Rightarrow \quad i = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

Definition of current

$$i = \frac{dq}{dt}$$

$$\Rightarrow \frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC} = -\frac{1}{RC}(q - C\mathcal{E})$$

↓ re-arrangement

$$\frac{dq}{q - C\mathcal{E}} = -\frac{dt}{RC}$$

↓ integration (use auxiliary variables q', t')

$$\int_0^q \frac{dq'}{q' - C\mathcal{E}} = -\int_0^t \frac{dt'}{RC}$$

$$\left[\ln(q' - C\mathcal{E})\right]_0^q = -\left[\frac{t'}{RC}\right]_0^t \Rightarrow \ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$

$$\Rightarrow \exp\left(\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right)\right) = \frac{q - C\mathcal{E}}{-C\mathcal{E}} = \exp\left(-\frac{t}{RC}\right)$$

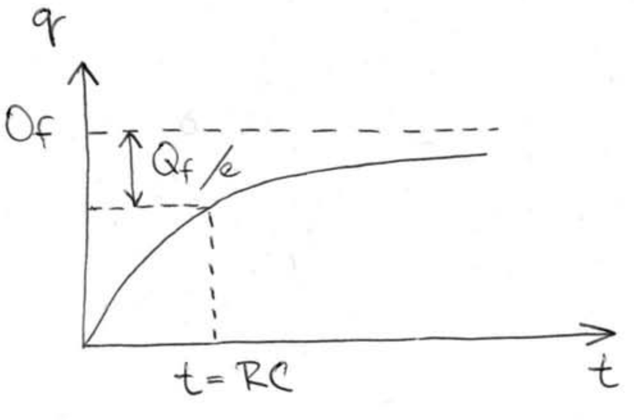
⇓

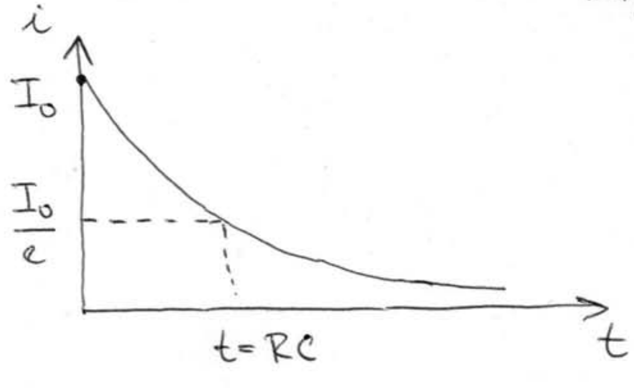
Instantaneous charge:

$$\boxed{q = C\mathcal{E}\left(1 - \exp\left(\frac{-t}{RC}\right)\right) = Q_f\left(1 - \exp\left(\frac{-t}{RC}\right)\right)}$$

Instantaneous current:

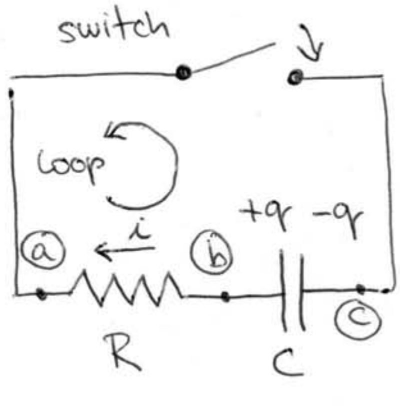
$$\boxed{i = \frac{dq}{dt} = \frac{\mathcal{E}}{R}\exp\left(-\frac{t}{RC}\right) = I_0 \exp\left(\frac{-t}{RC}\right)}$$





- q and i are exponential functions of time t
- Characteristic time scale of the capacitor charging process is given by the time constant of the circuit: $\boxed{\tau = R \cdot C}$
  - at time $t = \tau$, the charge has reached $\left(1 - \frac{1}{e}\right)$ of its final value Q_f and the current has dropped to $\frac{1}{e}$ of its initial value I_0
  - higher value of τ means slower charging → achieved by increasing R or C or both
  - both charge and current approach their final values asymptotically

Discharging a capacitor



- Initially, the capacitor is charged to Q_0 and the switch is open
- At time $t = 0$, the switch is closed → capacitor discharges through the resistor, its final charge drops to zero

- Kirchhoff's loop rule for capacitor discharging gives

$iR - \frac{q}{C} = 0$ and we have $i = \frac{-dq}{dt}$ (charge is decreasing with time so $\frac{dq}{dt} < 0$ but $i \cdot R$ should be positive)

$$-\frac{dq}{dt}\cdot R - \frac{q}{C} = 0 \longrightarrow \frac{dq}{dt} = -i = -\frac{q}{RC}$$

- At time $t=0$: $\boxed{i = I_0 = \frac{Q_0}{RC}}$ – initial current

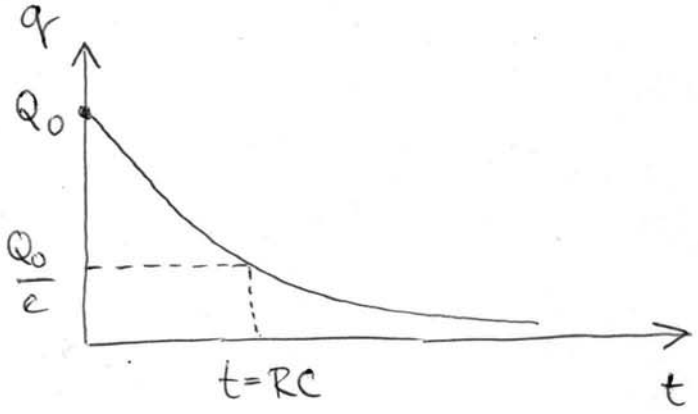
- Rearranging $\frac{dq}{dt} = \frac{-q}{RC}$ as $\frac{dq}{q} = \frac{-dt}{RC}$ and integrating

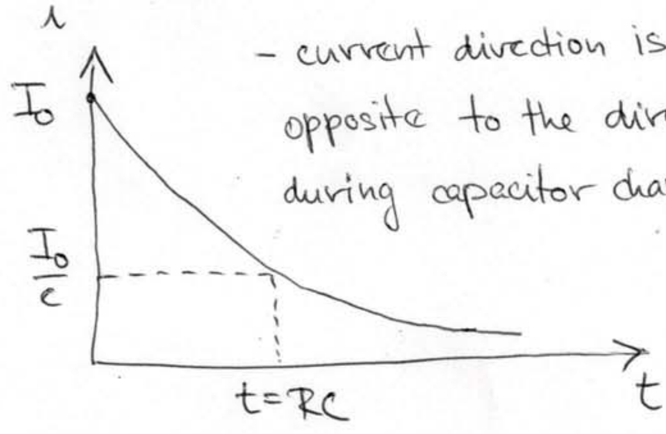
$$\int_{Q_0}^{q} \frac{dq'}{q'} = \int_0^t -\frac{dt'}{RC}\ ,\ \text{we obtain}\ \left[\ln(q')\right]_{Q_0}^{q} = \left[-\frac{t'}{RC}\right]_0^t$$

$$\ln\left(\frac{q}{Q_0}\right) = -\frac{t}{RC}$$

Instantaneous charge : $\boxed{q = Q_0 \exp\left(-\frac{t}{RC}\right)}$

Instantaneous current : $\boxed{i = -\frac{dq}{dt} = \frac{Q_0}{RC}\exp\left(\frac{-t}{RC}\right) = I_0 \exp\left(\frac{-t}{RC}\right)}$



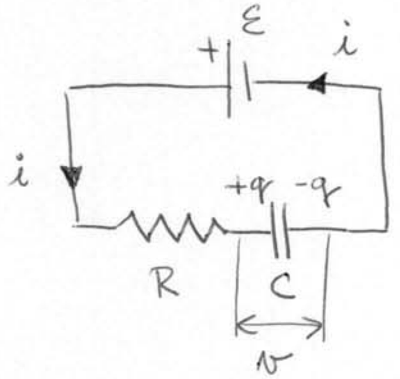


– current direction is now opposite to the direction during capacitor charging

- At time $t = RC$, both charge and current drop to $\frac{1}{e}$ of their initial values

Energy considerations

a) capacitor charging



From Kirchhoff's loop law

$$\mathcal{E} = iR + v \Rightarrow \mathcal{E}i = i^2R + vi$$

$\mathcal{E}i$: rate of energy delivered to the circuit by the battery

i^2R: rate of energy dissipation in the resistor

vi: rate of energy storage in the capacitor

Q_f ... final capacitor charge ($Q_f = \mathcal{E} \cdot C$)

V_f ... final capacitor voltage ($V_f = \mathcal{E}$)

- Total energy supplied by the battery during the capacitor charging:

$$W = \int_0^\infty \mathcal{E} i \, dt = \int_0^\infty \mathcal{E} \frac{dq}{dt} dt = \int_0^{Q_f} \mathcal{E} \, dq = \underline{\underline{\mathcal{E} Q_f}}$$

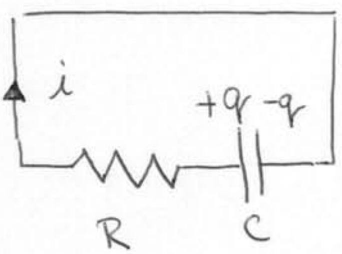
- Total energy stored in the capacitor:

$$U = \int_0^\infty v \cdot i \, dt = \int_0^\infty \frac{q}{C} \frac{dq}{dt} dt = \int_0^{Q_f} \frac{q}{C} dq = \frac{Q_f^2}{2C} =$$

$$= \frac{1}{2} V_f Q_f = \underline{\underline{\frac{1}{2} \mathcal{E} Q_f}}$$

$\Rightarrow$ half of the total supplied energy is stored in the capacitor, the other half is dissipated in the resistor $\rightarrow$ this result is independent of $\underline{\underline{R, C, \mathcal{E}}}$

b) capacitor discharging



Q_0 ... initial charge of the capacitor

From Kirchhoff's loop law

$$iR = -\frac{q}{C} \Rightarrow i^2R = -i\frac{q}{C}$$

i^2R: rate of energy dissipation in the resistor

$-i\frac{q}{C}$: rate of energy delivered to the circuit from the capacitor

- Total energy provided by the capacitor

$$U = \int_0^\infty -i\frac{q}{C}\,dt = \int_0^\infty -\frac{q}{C}\frac{dq}{dt}\,dt = \int_{Q_0}^{0} -\frac{q}{C}\,dq = +\frac{Q_0^2}{2C}$$

- Total energy dissipated in the resistor

$$W = \int_0^\infty i^2R\,dt = \int_0^\infty \frac{Q_0^2}{R^2C^2} e^{-\frac{2t}{RC}} \cdot R \cdot dt =$$

$$= \frac{Q_0^2}{RC^2}\int_0^\infty e^{-\frac{2t}{RC}}\,dt = \frac{Q_0^2}{RC^2}\left(-\frac{RC}{2}\right)\left[e^{-\frac{2t}{RC}}\right]_0^\infty =$$

$$= -\frac{Q_0^2}{2C}(0-1) = +\frac{Q_0^2}{2C}$$