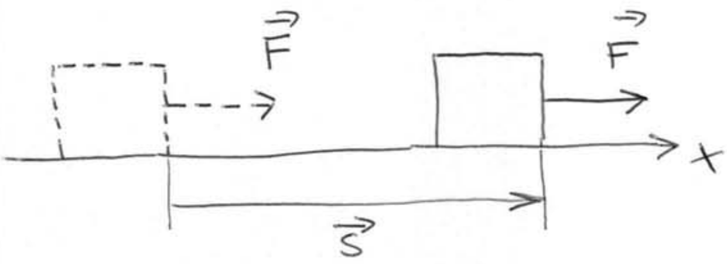
CHAPTER 6: WORK AND KINETIC ENERGY

- Principle of conservation of energy:

"Energy is a quantity that can be converted from one form to another but cannot be created or destroyed"

Section 6.1: Work



- Consider a body that undergoes a displacement $\vec{s}$ along a straight line under the action of a constant force $\vec{F}$ pointing in the same direction as the displacement $\vec{s}$

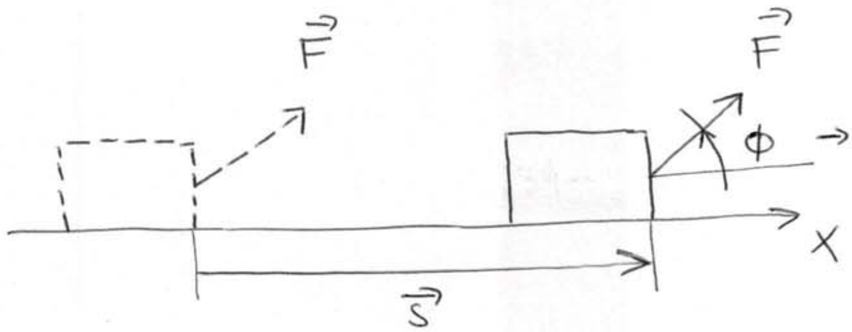
$\downarrow$

Definition of the work W done by the force $\vec{F}$ during the displacement:

$$\boxed{W = F \cdot s}$$

SI unit of work: $1 \text{ joule} \equiv 1\,J = 1\,N.m = 1\,\frac{kg.m^2}{s^2}$

• General case of a straight-line displacement under a constant force



Force components – parallel to the displacement : $F_{\parallel} = F\cos\phi$

– perpendicular to the displacement : $F_{\perp} = F \cdot \sin\phi$

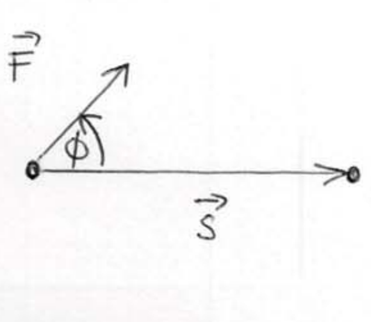
• Only the parallel component of the force $F_{\parallel}$ is contributing to the displacement $\vec{s}$

↓

Work $\boxed{W = F_{\parallel} \cdot s = F \cdot \cos\phi \cdot s = \vec{F} \cdot \vec{s}}$

⇒ for straight-line displacement under a constant force, the work is given by the scalar product of the force and displacement

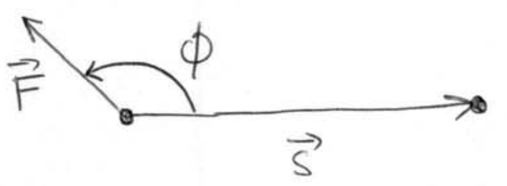
• Work done by a (constant) force can be positive, negative, or zero :



a) If the force has a component in the direction of displacement

$(0° \le \phi < 90°) \Rightarrow \cos\phi > 0$

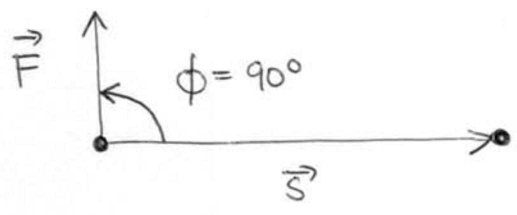
⇒ work done by the force is positive



b) If the force has a component opposite to the direction of displacement

$(90° < \phi \leq 180°) \Rightarrow \cos\phi < 0$

$\Rightarrow$ work done by the force is negative



c) If the force is perpendicular to the displacement

$\phi = 90° \Rightarrow \cos\phi = 0$

$\Rightarrow$ work done by the force is zero

EG
- When you hold a heavy object at a constant height above the earth's surface, you are not doing any work on the object (displacement $s = 0$)

- When you walk with the above object along the horizontal surface with constant velocity, you are still not doing any work on the object (your force is vertical but the displacement is horizontal $\Rightarrow \phi = 90° \Rightarrow \vec{F}\cdot\vec{s} = F\cdot s\cdot\cos\phi = 0$)

- When you lift the above object vertically from the earth's surface, you are doing positive work on the object (force and displacement have the same direction)

- When you lower the above object down to the earth's surface, you are doing negative work on the object (force and displacement have opposite directions)

- Newton's third law $\rightarrow$ when body A does positive work on body B, body B does an equal amount of negative work on body A

- When several forces act simultaneously on a body, the total work is

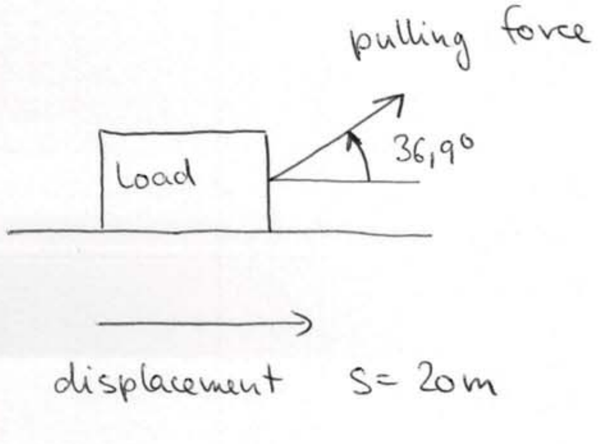
  a) the algebraic sum of the amounts of work done by the individual forces or

  b) the work done by the net force acting on the body

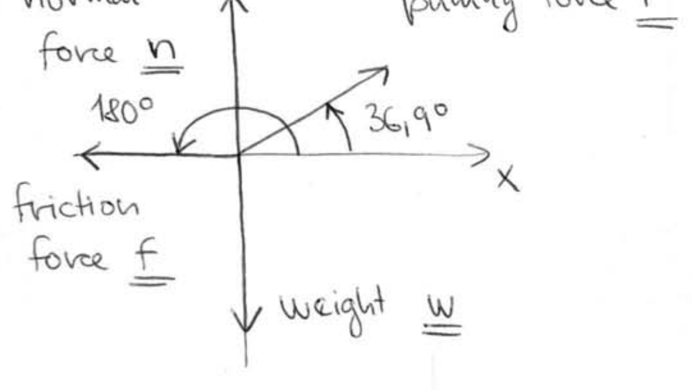
<u>Ex 6.2:</u> A farmer pulls a load with the total weight of 14.700 N along a distance of 20 m.

The applied pulling force has magnitude of 5000N and direction 36,9° above the horizontal. There is a 3500 N friction force opposing the motion of the load.

Find the work done by each force acting on the load and the total work of all forces.



Free-body diagram



Work of individual forces

$W_w = 0 \rightarrow$ weight and displacement are perpendicular

$W_n = 0 \rightarrow$ normal force and displacement are perpendicular

$$W_F = F \cdot s \cdot \cos\phi = 5000\,N \cdot 20\,m \cdot \cos 36{,}9° = 80000\,N \cdot m = \underline{\underline{80\,kJ}}$$

$$W_f = f \cdot s \cdot \cos\phi = 3500\,N \cdot 20\,m \cdot \cos 180° = -70\,000\,N \cdot m = \underline{\underline{-70\,kJ}}$$

$\downarrow$

Total work $W_{total} = W_w + W_n + W_F + W_f = 80\,kJ - 70\,kJ = \underline{\underline{10\,kJ}}$

Alternatively, we can calculate the net force parallel to the displacement $\sum F_x = F \cdot \cos 36{,}9° + f \cdot \cos 180° =$

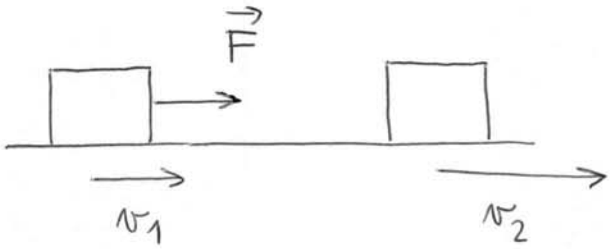
$$= 5000\,N \cdot \cos 36{,}9° - 3500\,N = \underline{\underline{500\,N}}$$

$\downarrow$

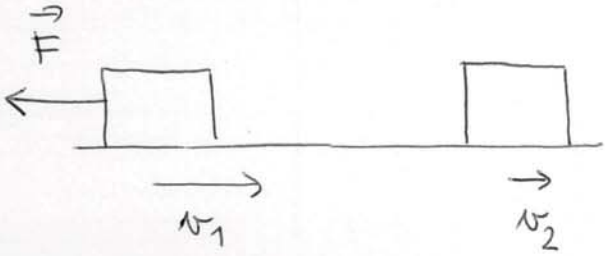
Total work $W_{total} = (\sum F_x) \cdot s = 500\,N \cdot 20\,m = 10\,000\,N \cdot m = \underline{\underline{10\,kJ}}$

Section 6.2: Kinetic energy and the work-energy theorem

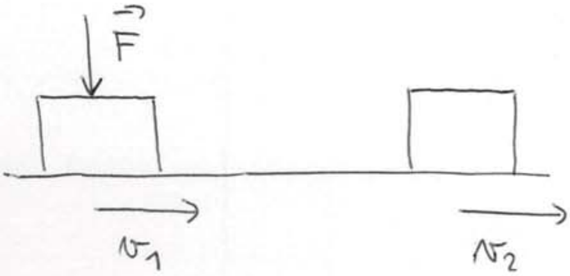
- The total work done on a body can be related to changes in the speed v of the body's motion



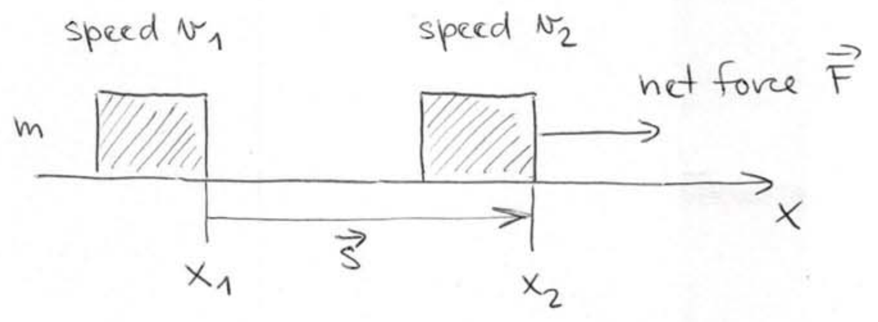
a) Net force $\vec{F}$ in the direction of body's motion
- positive work done on the body
- the speed of the body increases by Newton's 2nd law ($v_2 > v_1$)



b) Net force $\vec{F}$ against the direction of body's motion
- negative work done on the body
- the speed of the body decreases by Newton's 2nd law ($v_2 < v_1$)



c) Net force $\vec{F}$ perpendicular to the direction of body's motion
- zero work done on the body
- the speed of the body remains the same (no acceleration: $v_2 = v_1$)



Particle with mass $\underline{m}$ moving along the x-axis due to a constant net force $\underline{\vec{F}}$ directed along the positive x-axis

Newton's second law : $F = m \cdot a_x$

Constant acceleration motion ($t=0$ for particle located at $\underline{x_1}$):

$$\left.\begin{aligned} v_2 &= v_1 + a_x t \\ x_2 &= x_1 + v_1 t + \frac{a_x t^2}{2} \end{aligned}\right\} \quad t = \frac{v_2 - v_1}{a_x} \qquad (x_2 - x_1) = v_1 \left(\frac{v_2 - v_1}{a_x}\right) + \frac{a_x}{2} \frac{(v_2 - v_1)^2}{a_x^2}$$

$$\Downarrow$$

$$a_x = \frac{v_2^2 - v_1^2}{2(x_2 - x_1)} = \frac{v_2^2 - v_1^2}{2s}$$

Work done by force F :

$$\left[W = F \cdot s = m \cdot a_x \cdot s = m \cdot \frac{(v_2^2 - v_1^2)}{2s} \cdot s = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \right]$$

• Definition of the kinetic energy $\underline{K}$ of the moving particle:

$$\boxed{K = \frac{1}{2} m v^2}$$

– non-negative scalar quantity, equal to zero only when the particle is at rest

$$\Downarrow$$

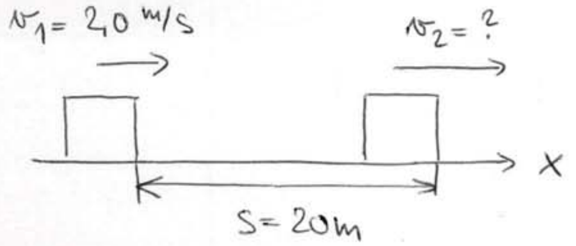
Work - energy theorem : " The total work done by the net force on a particle equals the change in the particle's kinetic energy "

$$\boxed{W_{tot} = K_2 - K_1 = \Delta K}$$

- If the work is
  a) positive, the kinetic energy increases
  b) negative, the kinetic energy decreases
  c) zero, the kinetic energy does not change

- From work-energy theorem, it follows that the units of work and kinetic energy are the same : $1\,J = 1\,\frac{kg\,m^2}{s^2}$

<u>Ex 6.3:</u> An object with total weight 14.700 N moves horizontally with initial speed $v_1 = 2{,}0\,\frac{m}{s}$. Over the distance of 20m, the total work done on the object is 10 kJ. What is the speed of the object after it moves 20 m?



Object weight $w = m \cdot g$

$\Downarrow$

Object mass $m = \frac{w}{g} = \frac{14700\,N}{9{,}80\,\frac{m}{s^2}} = \underline{1500\,kg}$

Initial kinetic energy : $K_1 = \frac{1}{2} m v_1^2 = \frac{1}{2}\,1500\,kg\,\left(2\,\frac{m}{s}\right)^2 = \underline{3000\,J}$

Final kinetic energy : $K_2 = \frac{1}{2} m v_2^2 = K_1 + W_{tot} = 13000\,J$

$\Downarrow$

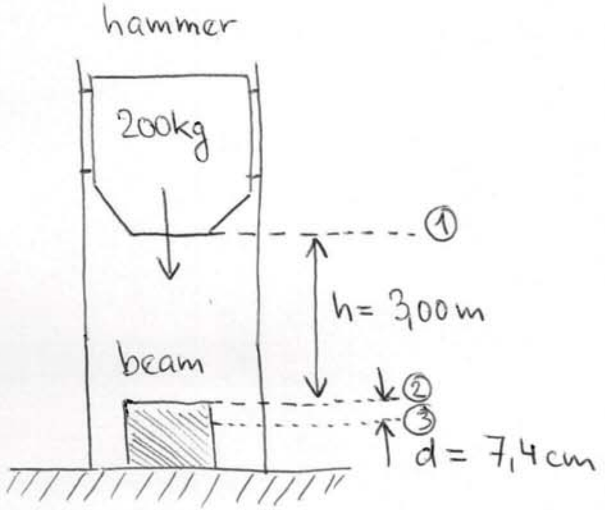
$$v_2 = \sqrt{\frac{2(K_1 + W_{tot})}{m}} = \sqrt{\frac{2 \cdot 13000\,J}{1500\,kg}} = \underline{4{,}2\,\frac{m}{s}}$$

After moving 20 meters, the speed of the object increases to $4{,}2\,\frac{m}{s}$ as a result of the work done on the object.

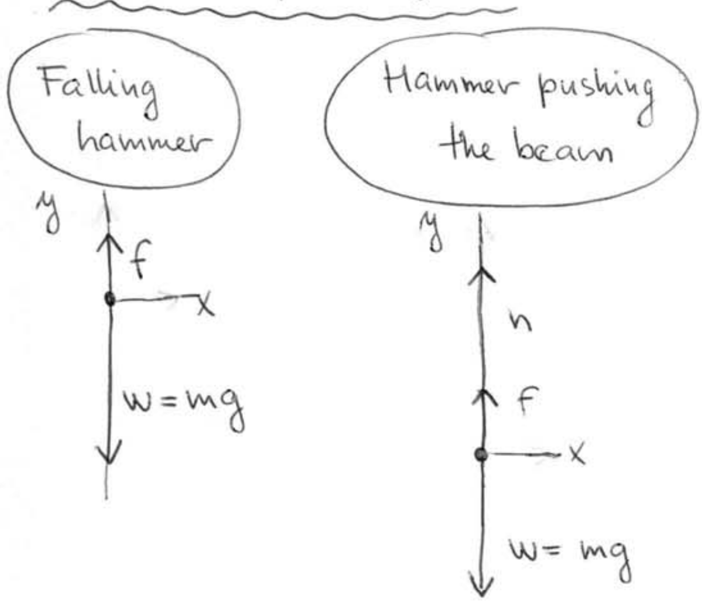
Ex 6.4: Steel hammer with mass 200 kg is lifted 3,00 m above a vertical beam being driven into the ground (see picture). When the hammer is dropped, it drives the beam 7,4 cm into the ground. During the hammer motion, a constant 60 N friction force acts on it.

Find a) the speed of the hammer when it hits the beam

b) the average force of the hammer on the beam



Free-body diagrams



$w = mg$ ... hammer weight

$f = 60\,N$ ... friction force

n ... normal force of the beam on the hammer (assumed to be constant)

ad a)

Falling hammer: net vertical force $\sum F_y = f + (-mg) = 60N - (200 \times 9{,}8)N$

$= -1900\,N$

net vertical displacement $s = -h = -3{,}00\,m$

$\Downarrow$

work done on the hammer during falling

$W_{tot} = (\sum F_y) \cdot s = (-1900N) \cdot (-3\,m) = \underline{\underline{5700\,J}}$

Work-energy theorem

$$W_{tot} = K_2 - K_1 = K_2 - 0 = \frac{1}{2} m v_2^2 \Rightarrow v_2 = \sqrt{\frac{2W_{tot}}{m}} = \sqrt{\frac{2 \cdot 5700\,J}{200\,kg}} = 7{,}55\,\frac{m}{s}$$

hammer is initially at rest

The speed of the hammer when it hits the beam is $7{,}55\,\frac{m}{s}$.

(ad b) Hammer pushing the beam

net vertical force $\sum F_y = n + f + (-mg)$

vertical displacement $s = -d = -0{,}074\,m$

$\Downarrow$

work done on the hammer during pushing the beam

$$W_{tot} = \left(\sum F_y\right) s = (n + f - mg) \cdot (-d)$$

and also

$$W_{tot} = K_3 - K_2 = 0 - \frac{1}{2} m v_2^2$$

hammer comes to stop after pushing the beam

$\Downarrow$

$$(n + f - mg)(-d) = -\frac{1}{2} m v_2^2$$

$$n = \frac{m v_2^2}{2d} + mg - f = \frac{5700\,J}{0{,}074\,m} + (200 \times 9{,}8)\,N - 60\,N$$

$$= \underline{\underline{79\,000\,N}}$$

From Newton's third law, the magnitude of the average force of the hammer on the beam is equal to the magnitude of the average force of the beam on the hammer, i.e. 79 000 N which is 40x more than the hammer's weight.

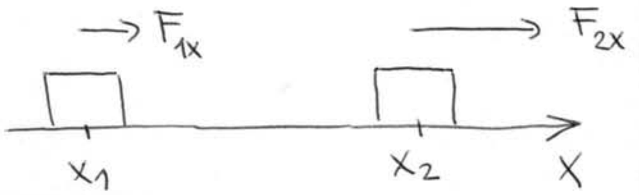
NOTE 1: From work-energy theorem, it follows that the kinetic energy of a particle is equal to the total work that was done to accelerate the particle from rest to its present speed.

Analogically, the kinetic energy of a particle is equal to the total work the particle can do in the process of being brought to rest.

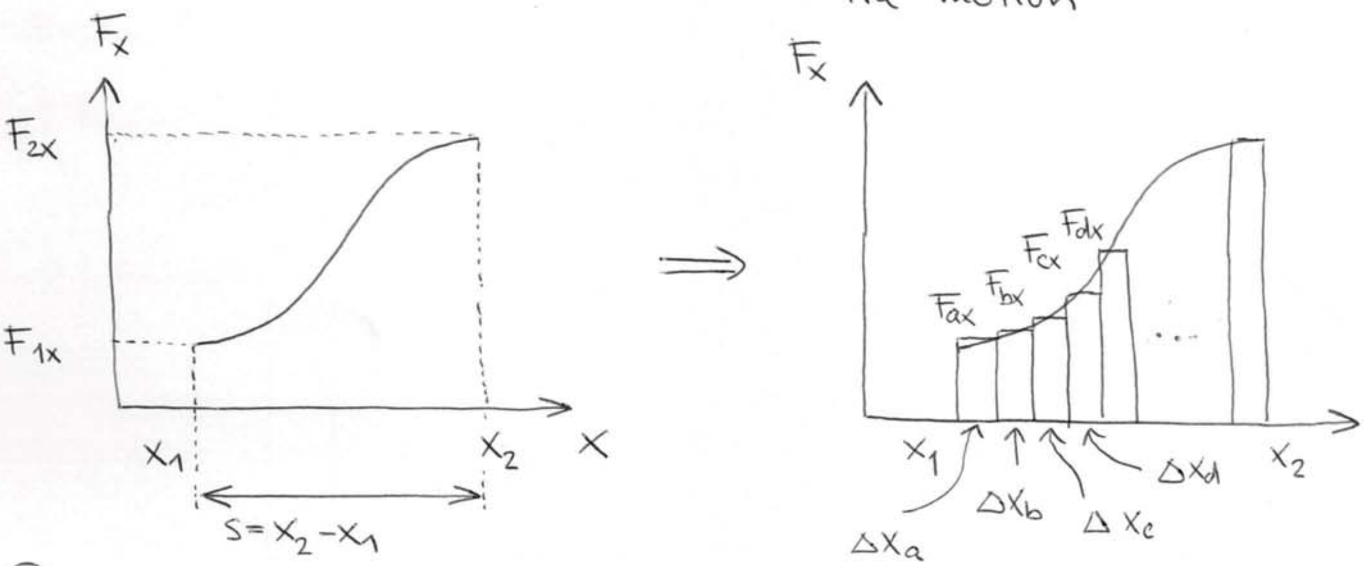
NOTE 2: In composite systems with internal degrees of freedom that cannot be represented as a single moving particle, the total kinetic energy of the system can change even if there is no work done on the system by external forces.

Section 6.3: Work and energy with varying forces

Work done by a varying force in straight-line motion



Suppose a particle moves along the x-axis from point x_1 to x_2 under a force whose x-component changes during the motion



(?) What is the total work done by the varying force $F_x(x)$ during displacement from x_1 to x_2?

↳ Let's divide the total displacement $s = x_2 - x_1$ into small segments $\Delta x_a, \Delta x_b, \Delta x_c \ldots$

↓

Work done by the force during segment Δx_a can be approximated as the average Force F_{ax} in that segment multiplied by the x- displacement Δx_a $(W_a = F_{ax} \Delta x_a)$

- Applying the above procedure to all segments gives the total approximate work done by the force in the total displacement from x_1 to x_2

$$W \approx F_{ax}\Delta x_a + F_{bx}\Delta x_b + \dots$$

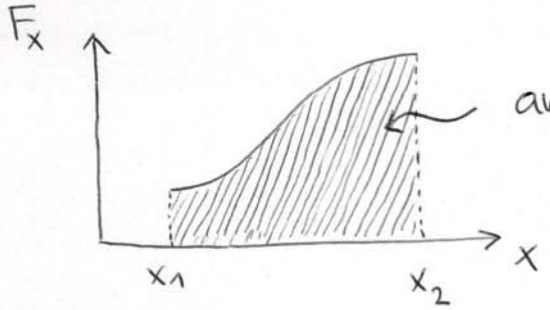
$\downarrow$

If the number of segments is very large, $\Delta x \to 0$ and we obtain the exact formula for the total work

$$\boxed{W = \int_{x_1}^{x_2} F_x\, dx}$$

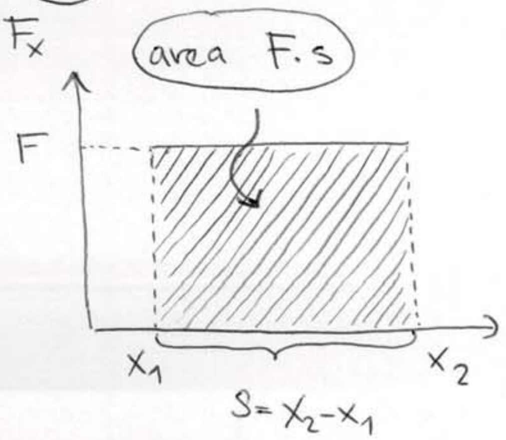
Total work = integral of F_x from x_1 to x_2

- Total work done by the force corresponds to the area under the graph of force as a function of position between the initial and final positions



$$\text{area} = \int_{x_1}^{x_2} F_x\, dx = W$$

(EG) If the force F_x is constant, it can be taken out of the integral

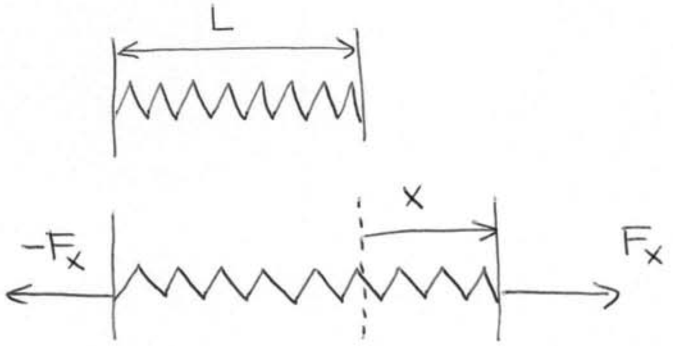


$$W = \int_{x_1}^{x_2} F.\,dx = F\int_{x_1}^{x_2} dx = F(x_2 - x_1)$$

$$= F.s$$

Stretched spring

- Consider a spring with unstretched length $\underline{\underline{L}}$



To keep the spring stretched beyond its unstretched length $\underline{\underline{L}}$ by an amount $\underline{\underline{x}}$, force $\underline{\underline{F_x}}$ has to be applied at each end

— for small elongations of the spring, the force needed to stretch the spring is directly proportional to $\underline{\underline{x}}$

$$\boxed{F_x = k \cdot x}$$

Hooke's law

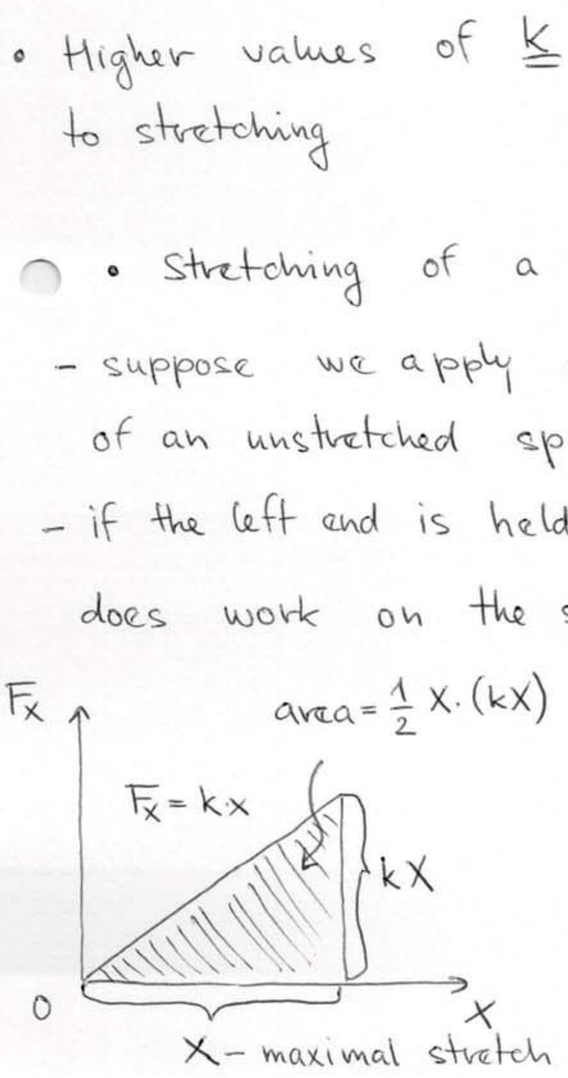
k ... force constant (spring constant)

$[k] = \frac{N}{m}$

- Higher values of $\underline{\underline{k}}$ correspond to higher resistance to stretching

- Stretching of a spring requires work

- suppose we apply equal and opposite forces to the ends of an unstretched spring and gradually increase the forces
- if the left end is held stationary, only the right-end force does work on the spring during the stretching



$$W = \int_0^X F_x \, dx = \int_0^X kx \, dx = \left[\tfrac{1}{2} kx^2 \right]_0^X$$

$$= \frac{1}{2} k X^2$$

Corresponds to the area of the shaded triangle

→ the work required to stretch an originally unstretched spring by an amount $\underline{\underline{X}}$ depends quadratically on $\underline{\underline{X}}$

- if the spring was initially stretched by a distance $\underline{\underline{x_1}}$ and the final stretching distance is $\underline{\underline{x_2}}$, the work needed for additional elongation is

$$W = \int_{x_1}^{x_2} F_x \, dx = \int_{x_1}^{x_2} kx \, dx = \left[\frac{kx^2}{2} \right]_{x_1}^{x_2} = \frac{1}{2} k x_2^2 - \frac{1}{2} k x_1^2$$

- Compression of a spring can be described identically to the spring stretching → in order to compress a spring by an amount $\underline{\underline{X}}$, positive work $\frac{1}{2} kX^2$ is needed

Ex 6.6: A woman with weight 600N steps on a scale containing a spring. In equilibrium, the spring is compressed by 1cm under her weight. Find the force constant and the total work done on the spring during compression

Equilibrium: weight $w =$ spring force $k \cdot x \Rightarrow k = \frac{w}{x} = \frac{600\,N}{0{,}01\,m}$

$$= \underline{\underline{6 \times 10^4 \frac{N}{m}}}$$

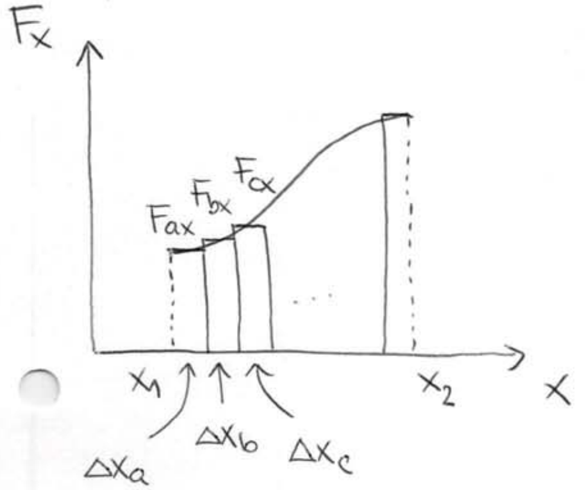
Total work done on the spring during compression

$$W = \frac{1}{2} kx^2 = \frac{1}{2} \, 6 \times 10^4 \frac{N}{m} \cdot (0{,}01\,m)^2 = \underline{\underline{3{,}0\,J}}$$

Positive work is needed to compress the spring

Work-energy theorem for straight-line motion with varying forces

- Let's consider a particle that undergoes a straight-line displacement x under a net force with varying x-component F_x



Let's divide the total displacement into small segments $\Delta x_a, \Delta x_b, \dots$

$\downarrow$

We can apply the work-energy theorem to each segment because the value of F_x in each segment is approximately constant

$$\left.\begin{aligned} W_a &= K_a - K_1 \\ W_b &= K_b - K_a \\ &\vdots \\ W_y &= K_y - K_x \\ W_z &= K_2 - K_y \end{aligned}\right\}$$

Total work done on the particle is

$$W_{tot} = W_a + W_b + \dots W_y + W_z =$$
$$= K_a - K_1 + K_b - K_a + \dots + K_y - K_x + K_2 - K_y$$
$$= K_2 - K_1$$

$$\boxed{W_{tot} = K_2 - K_1} \Longrightarrow$$ Work-energy theorem also holds for straight-line motion with varying force

- Alternatively, work-energy theorem for varying forces can be derived from Newton's second law

NOTE acceleration $a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx} \underbrace{\frac{dx}{dt}}_{v_x} = v_x \frac{dv_x}{dx}$

$$W_{tot} = \int_{x_1}^{x_2} F_x \, dx \underset{\text{2nd Law}}{=} \int_{x_1}^{x_2} m a_x \, dx = \int_{x_1}^{x_2} m v_x \frac{dv_x}{dx} \cdot dx$$

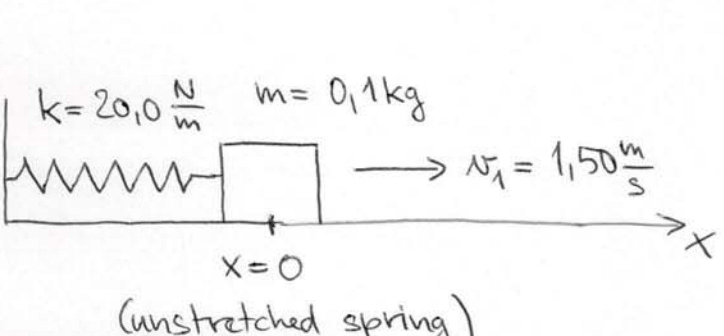
- realizing that $dv_x = \left(\frac{dv_x}{dx}\right) dx$, we can change integration variable from $\underline{\underline{x}}$ to $\underline{\underline{v_x}}$

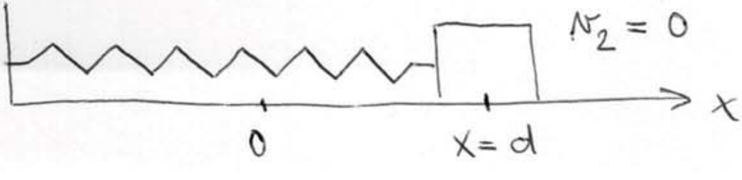
$$\left[W_{tot} = \int_{v_1}^{v_2} m v_x \, dv_x = \left[\frac{m v_x^2}{2} \right]_{v_1}^{v_2} = \frac{m v_2^2}{2} - \frac{m v_1^2}{2} \right]$$

↳ we obtained again the work-energy theorem

<u>Ex 6.7:</u> A particle with mass 0,1 kg is attached to the end of a horizontal spring with force constant 20,0 $\frac{N}{m}$.
The spring is originally unstretched and the particle is moving at 1,50 $\frac{m}{s}$ to the right. Find the maximum distance $\underline{\underline{d}}$ that the particle moves to the right

a) if there is no friction
b) if there is kinetic friction coefficient $\mu_k = 0{,}47$

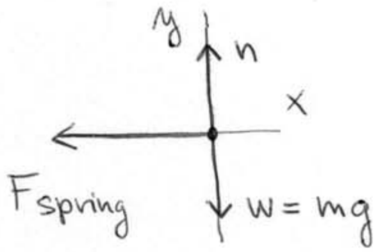




→ maximal stretch of the spring
↓
particle is at rest

ad a)

Free- body diagram



Motion is purely horizontal

$\downarrow$

The glider moving from $x=0$ to $x=d$ does work W on the spring given by

$$W_s = \frac{1}{2}kd^2 - \frac{1}{2}k(0)^2 = \frac{1}{2}kd^2$$

$\downarrow$

The spring does work $W_G = -W_S = -\frac{1}{2}kd^2$ on the glider.

From work - energy theorem

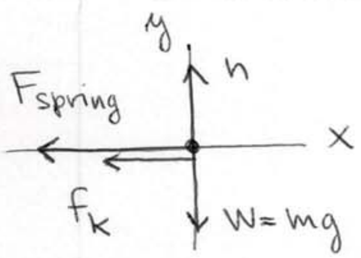
$$W_G = K_2 - K_1 \implies -\frac{1}{2}kd^2 = 0 - \frac{1}{2}mv_1^2$$

$\Downarrow$

$$d = v_1\sqrt{\frac{m}{k}} = 1{,}50\,\frac{m}{s}\sqrt{\frac{0{,}1\,kg}{20\,N/m}} = 0{,}106\,m = \underline{\underline{10{,}6\,cm}}$$

ad b)

Free -body diagram



Friction force

$f_k = \mu_k mg$

In addition to work $W_G = -\frac{1}{2}kd^2$ done on the particle by the spring, there is work done on the particle by the friction: $W_{fric} = f_k d \cdot \cos 180^\circ =$

$= -\mu_k mgd$

$\Downarrow$

Total work done on the spring is $W_{tot} = W_G + W_{fric} =$

$$= -\frac{1}{2}kd^2 - \mu_k mgd$$

From work-energy theorem : $W_{tot} = K_2 - K_1$

$$\Downarrow$$

$$-\frac{1}{2}kd^2 - \mu_k mgd = 0 - \frac{1}{2}mv_1^2$$

$$\frac{1}{2}kd^2 + \mu_k mgd - \frac{1}{2}mv_1^2 = 0 \rightarrow \text{quadratic equation for } \underline{\underline{d}}$$

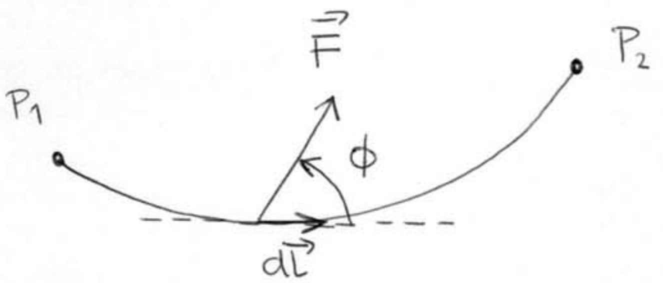
$$d = \frac{-\mu_k mg \pm \sqrt{(\mu_k mg)^2 + kmv_1^2}}{k} = \dots = \begin{cases} 0{,}086\,m \\ -0{,}132\,m \end{cases}$$

We can discard the negative root $\Rightarrow$ $d = 0{,}086\,m = \underline{\underline{8{,}6\,cm}}$

With friction, particle will move a shorter distance before coming to stop.

Work- energy theorem for motion along a curve

- Suppose a particle moves from point $\underline{\underline{P_1}}$ to $\underline{\underline{P_2}}$ along a curve under the action of a varying force $\underline{\underline{\vec{F}}}$



Curved path P_1P_2 can be divided into many infinitesimal displacements $\underline{\underline{d\vec{l}}}$, each of them tangent to the path

ϕ ... angle between the force $\vec{F}$ and the displacement $d\vec{l}$

$\rightarrow$ element of work done during the displacement $\underline{\underline{d\vec{l}}}$ is

$$dW = F.\cos\phi\, dl = F_{\parallel}\, dl = \vec{F}\cdot d\vec{l}$$

→ the total work done by $\vec{F}$ along the whole path P_1P_2 is

$$\boxed{W = \int_{P_1}^{P_2} dW = \int_{P_1}^{P_2} F \cdot \cos\phi \, dl = \int_{P_1}^{P_2} F_{\parallel} \, dl = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}}$$

Line integral of force $\vec{F}$ along the path P_1P_2

- Assuming the Force $\vec{F}$ is essentially constant over any segment $d\vec{l}$, we can apply the work-energy theorem to that segment

↓

- adding up elements of work and changes of the kinetic energy for all segments along the whole path leads to the work-energy theorem for the whole path

$$\boxed{W_{tot} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} = K_2 - K_1}$$

- valid for arbitrary path and character of the force

- Only the component $F_{\parallel}$ of the net force parallel to the path does work on the particle and can change the particle kinetic energy / speed → perpendicular component $F_{\perp}$ only changes direction of the particle's motion

NOTE: $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ and $d\vec{l} = dx \hat{i} + dy \hat{j} + dz \hat{k}$

⇓

$$W_{tot} = \int_{P_1}^{P_2} \vec{F} d\vec{l} = \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

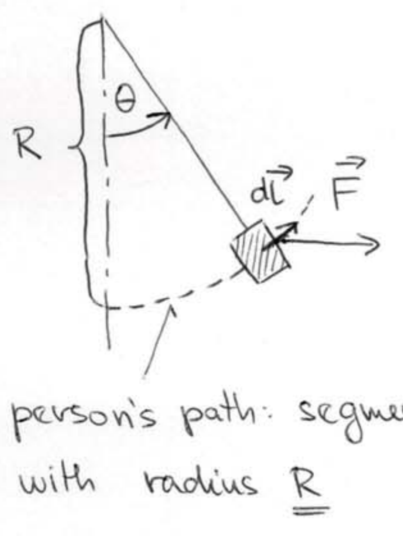
→ total work can be calculated using the components of the force and displacement vectors

Ex 6.8: A person with weight w is pushed in a swing with a varying horizontal force $\vec{F}$ increasing such that the person and the swing move very slowly and remain nearly in equilibrium. The length of the swing chains is R and they make the maximal angle with the vertical of θ_0. Motion starts with the angle $\theta = 0$.

a) What is the total work done on the person by all forces?

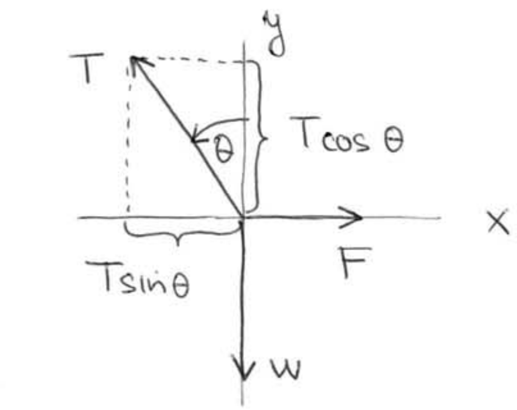
b) What is the work done by the tension T in the chains?

c) What is the work done by the horizontal pushing force $\vec{F}$?



person's path: segment of a circle with radius R

Free-body diagram



(ad a) The person remains in equilibrium at every point

$\Rightarrow \sum \vec{F} = 0 \Rightarrow$ speed is constant

$$W_{tot} = \int_{path} \left(\sum \vec{F}\right) d\vec{l} = 0$$

(ad b) Tension $\underline{\underline{T}}$ is perpendicular to displacement $\underline{\underline{\vec{dl}}}$ at all points on the person's path

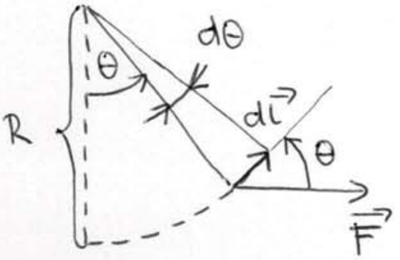
↓

$$W_T = \int_{path} T \cdot \cos\phi \cdot dl = \int_{path} T \cdot \cos 90^\circ \, dl = \underline{\underline{0}}$$

(ad c) Person is in equilibrium:

$$\left.\begin{aligned} \sum F_x &= F + (-T \cdot \sin\theta) = 0 \\ \sum F_y &= T \cdot \cos\theta + (-w) = 0 \end{aligned}\right\} \quad T = \frac{w}{\cos\theta}$$

$$F = T \cdot \sin\theta = w \frac{\sin\theta}{\cos\theta} = \underline{\underline{w \tan\theta}}$$



Person is moving on a circle of radius $\underline{\underline{R}}$ $\rightarrow$ $|\vec{dl}| = R\,d\theta$

Angle between $\underline{\underline{\vec{F}}}$ and $\underline{\underline{\vec{dl}}}$ is $\underline{\underline{\theta}}$

↓

$$W_F = \int_{P_1}^{P_2} F_{\parallel} \, dl = \int_0^{\theta_0} F \cdot \cos\theta \, dl = \int_0^{\theta_0} w \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos}\,\theta \cdot R\,d\theta =$$

$$= wR \int_0^{\theta_0} \sin\theta \, d\theta = wR \left[-\cos\theta\right]_0^{\theta_0} = \underline{\underline{wR(1-\cos\theta_0)}}$$

Alternatively, we can use the components of the force and displacement

$$\left.\begin{aligned} \vec{F} &= F \cdot \hat{i} = w \tan\theta \, \hat{i} \\ \vec{dl} &= R\,d\theta \cos\theta \, \hat{i} + R\,d\theta \sin\theta \, \hat{j} \end{aligned}\right\} \quad W_F = \int_0^{\theta_0} \vec{F} \cdot \vec{dl} = \int_0^{\theta_0} \Big(w \tan\theta \, R\,d\theta \cos\theta$$

$$+ 0 \cdot R\,d\theta \sin\theta \Big) = wR \int_0^{\theta_0} \frac{\sin\theta}{\cos\theta} \cos\theta \, d\theta$$

$$= wR \int_0^{\theta_0} \sin\theta \, d\theta \quad \text{as before}$$

Section 6.4: Power

- Power is the time rate at which work is done

$\downarrow$

When amount of work ΔW is done during a time interval Δt, the average power is defined as

$$\boxed{P_{av} = \frac{\Delta W}{\Delta t}}$$

- Taking the limit of average power for time interval $\Delta t \to 0$ gives instantaneous power

$$\boxed{P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}}$$ - derivative of work with respect to time

SI unit of power: $1 \text{ watt} \equiv 1 W = 1 \frac{J}{s}$

- Power can be also expressed in terms of force and velocity

$\downarrow$

Suppose a (constant) force $\vec{F}$ acts on a body undergoing a vector displacement $\Delta \vec{s}$ in time interval Δt

$\downarrow$

The work done by the force is $\Delta W = \vec{F} \cdot \Delta \vec{s}$ and the average power is then $P_{av} = \frac{\Delta W}{\Delta t} = \vec{F} \cdot \frac{\Delta \vec{s}}{\Delta t} = \vec{F} \cdot \vec{v}_{av}$

Taking the limit $\Delta t \to 0$ gives the instantaneous power as

$$\boxed{P = \vec{F} \cdot \vec{v}}$$

Ex 6.11: A 50 kg runner runs to the top of 443 m tall building in 15 minutes. What is her average power output required to climb the building?

To move a 50 kg runner 443 m against gravity, work

$W = mgh = 50\,\text{kg} \cdot 9{,}80\,\frac{\text{m}}{\text{s}^2} \cdot 443\,\text{m} = 2{,}17 \times 10^5\,\text{J}$ has

to be done.

If the work is done in 15 minutes (= 900 s), the power output is

$$P = \frac{W}{t} = \frac{2{,}17 \times 10^5\,\text{J}}{900\,\text{s}} = \underline{\underline{241\,\text{W}}}$$