CHAPTER 11: EQUILIBRIUM AND ELASTICITY

Equilibrium = state in which a body does not have any linear or angular acceleration in an inertial frame of reference

Section 11.1: Conditions for equilibrium

- First Newton's law – a particle is in equilibrium (zero linear acceleration) in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero

$$\downarrow$$

For an extended body: equilibrium describes the state in which the center of mass of the body has zero acceleration

→ the vector sum of all external forces acting on the body is zero

$$\Downarrow$$

First condition for equilibrium

$$\boxed{\sum \vec{F} = 0} \quad \text{or} \quad \boxed{\sum F_x = 0,\ \sum F_y = 0,\ \sum F_z = 0}$$

- Equilibrium of an extended body additionally requires no tendency of the body to rotate in an inertial frame

$$\downarrow$$

- zero angular momentum of the body about any point and zero rate of change of angular momentum about any point

- Angular momentum is related to torque by : $\underbrace{\sum \vec{\tau}}_{\text{net torque of external forces}} = \underbrace{\frac{d\vec{L}}{dt}}_{\text{rate of change of angular momentum}}$

$$\Downarrow$$

Second condition for equilibrium

$$\boxed{\sum \vec{\tau} = 0}$$

" The sum of the torques due to all external forces acting on the body, with respect to any specified point, must be zero "

- Static equilibrium of a rigid body
  $\longrightarrow$ body is at rest (no translation or rotation)

- General (non-static) equilibrium also permits uniform translational motion without rotation

(EG) Car going on a straight, level road with a constant speed

Section 11.2: Center of gravity

- In a typical equilibrium problem, we need to calculate the torque of body's weight which is distributed over the entire body (gravity acting on each element of the body)

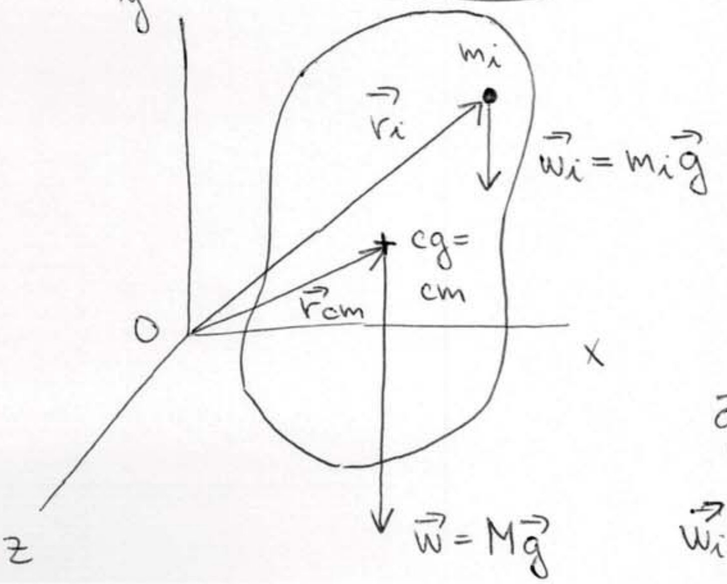
$\downarrow$

Alternatively, we can assume that the entire force of gravity is concentrated at a single point called the center of gravity ("cg")

- Assuming constant acceleration due to gravity $\vec{g}$ over the entire body, the center of gravity is identical to the center of mass of the body ("cm")

- Consider a body made of particles with masses m_i, located at points with position vectors $\vec{r}_i$ with respect to origin O

total mass $M = \sum_i m_i$



Definition of the center of mass

$$\boxed{\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}}$$

$\vec{g}$ ... constant acceleration due to gravity

$\Downarrow$

$\vec{w}_i = m_i\vec{g}$ ... weight of i-th particle

$\vec{w} = M\vec{g} = \left(\sum_i m_i\right)\vec{g}$ ... total weight

- The torque $\vec{\tau}_i$ of the weight $\vec{w}_i$ of i-th particle with respect to origin O is

$$\vec{\tau}_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i \vec{g}$$

$$\Downarrow$$

The total torque due to the gravitational forces on all particles is

$$\vec{\tau} = \sum_i \vec{\tau}_i = \sum_i (\vec{r}_i \times m_i \vec{g}) = \vec{r}_1 \times m_1 \vec{g} + \vec{r}_2 \times m_2 \vec{g} + \dots$$

$$= (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots) \times \vec{g} = \left(\sum_i m_i \vec{r}_i\right) \times \vec{g}$$

$$\Downarrow$$

divide and multiply the right-hand side by the total mass M

$$\vec{\tau} = \frac{\left(\sum_i m_i \vec{r}_i\right)}{M} \times M\vec{g} = \left(\frac{\left(\sum_i m_i \vec{r}_i\right)}{\sum_i m_i}\right) \times (M\vec{g})$$

$\vec{w}$ ... total weight

$\vec{r}_{cm}$ ... center of mass

$$\Downarrow$$

$$\boxed{\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w}}$$

The total gravitational torque is equal to the torque of total weight of the body $\vec{w}$ acting at the position $\vec{r}_{cm}$

(for uniform value of $\vec{g}$ over the entire body)

$$\downarrow$$

center of gravity "cg" $\Longleftrightarrow$ center of mass "cm"

- For varying $\vec{g}$, the two points are separated from each other

- Symmetric, homogeneous objects (sphere, cube)
  $\rightarrow$ the center of gravity corresponds to the geometric center

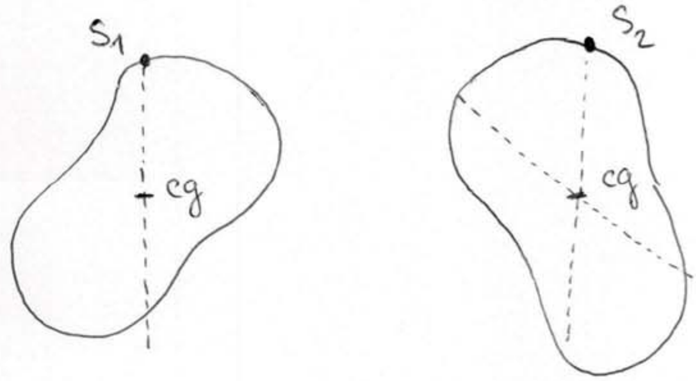
- Center of gravity of a system of bodies with masses $\underline{\underline{m_i}}$ and centers of gravity located at positions $\underline{\underline{\vec{r}_i}}$ is given by

$$\vec{r}_{cg,system} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

- Consider a body acted on by gravity and suspended at a single point

$\downarrow$

In equilibrium, the center of gravity is always at or directly above/below the point of suspension so that the weight has zero torque about the suspension point

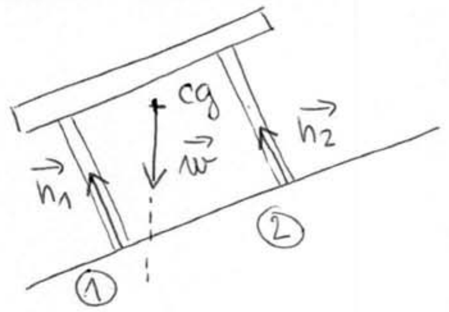


$S_{1,2}$ ... suspension points
cg ... center of gravity

$\downarrow$

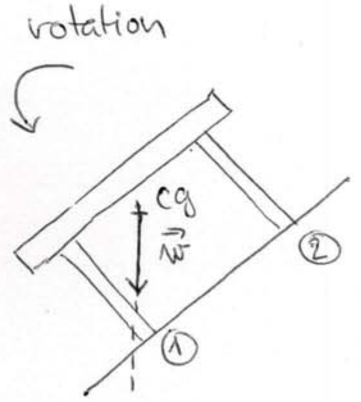
located at the intersection of vertical lines through suspension points S_1, S_2

- A body supported at several points must have its center of gravity somewhere within the area bounded by the supports



Torque of weight $\underline{\vec{w}}$ with respect to point ① is opposite to the torque of normal force $\underline{\vec{n_2}}$ with respect to ①

→ no rotation

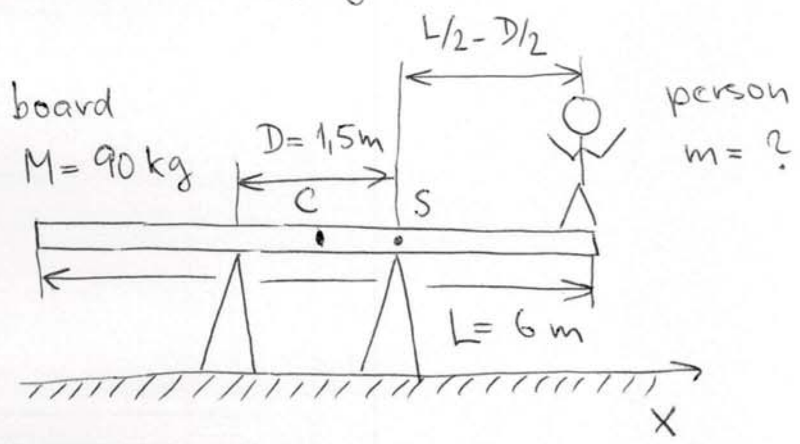


Torque of weight $\underline{\vec{w}}$ with respect to point ① is not compensated by the torque of normal forces

→ body tips over and rotates

<u>Ex 11.1</u>: A uniform board of length $L = 6{,}0\,m$ and mass $M = 90\,kg$ is supported as shown in the picture.

The separation distance of the supports is $D = 1{,}5\,m$ and they are located symmetrically relative to the center of the board. What is the maximal mass of a person that can stand on the end of the board without tipping it over?



C… center of gravity of the board (the origin of coordinate system)

S… maximal safe position of the system's center of gravity

- Maximal safe distance of the system's center of gravity from the origin is $D/2$ → if the system's center of gravity moves to the right of the right support, the board will tip over

Position of the center of gravity: $x_{cg} = \dfrac{M\cdot(0) + m\cdot\left(\frac{L}{2}\right)}{M+m} = \dfrac{mL}{2(M+m)}$

$$x_{cg}^{max} = \frac{D}{2} \quad \rightarrow \quad \frac{mL}{2(M+m)} = \frac{D}{2}$$

$$mL = MD + mD \Rightarrow m = \frac{MD}{(L-D)} =$$

$$= \left(\frac{90 \cdot 1{,}5}{6 - 1{,}5}\right) kg = \underline{30\,kg}$$

The board can provide a stable support to a person with a mass no more than 30 kg.

Section 11.3: Solving rigid-body equilibrium problems

- Two conditions have to be fulfilled simultaneously for a rigid body in equilibrium (zero acceleration)

$$\sum \vec{F} = 0 \quad \text{and} \quad \sum \vec{\tau} = 0$$

- For a simplified situation of forces $\underline{F}$ lying in the xy-plane (i.e. $F_z = 0$ for all forces), torques $\underline{\tau}$ have only z-component perpendicular to the plane

$$\Rightarrow \sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \quad \text{and} \quad \sum \tau_z = 0$$

- Reference point for torque calculation can be chosen arbitrarily → we can select it so as to simplify the calculations as much as possible

↓

- forces with force lines passing through the selected reference point have zero torques about that point

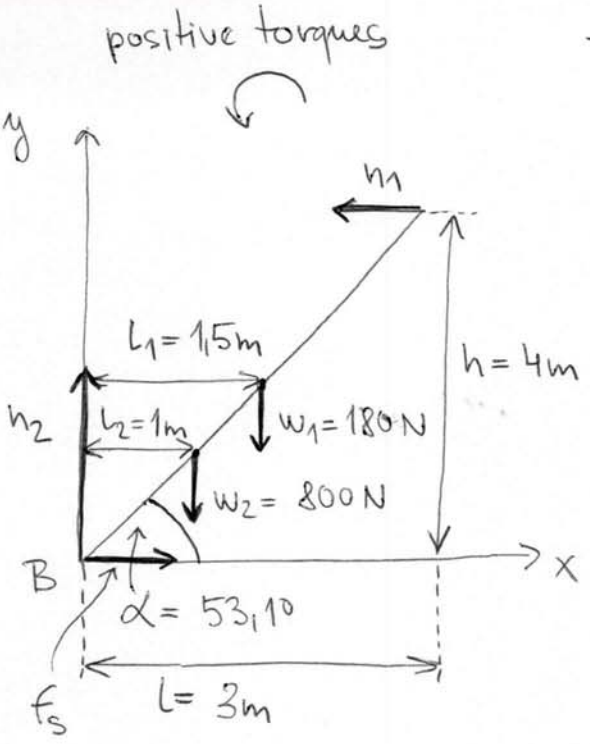
NOTE: reference point does not have to represent an actual axis of rotation

Ex 11.3: A person with weight 800N climbs a uniform ladder that is 5,0m long and weights 180 N. The person is standing one-third of the way up the ladder (see the picture). The bottom of the ladder rests on a horizontal surface with a static friction coefficient μ_s and the top of the ladder rests on a frictionless wall. The ladder makes an angle of 53,1° with the horizontal, forming a 3-4-5 right-angle triangle.

a) Find the normal and friction forces at the base of the ladder

b) Find the minimal static friction coefficient needed to prevent ladder slipping at the base

c) Find the magnitude and direction of the contact force on the ladder at the base



n_1 ... normal force of the wall at the top of the ladder

n_2 ... normal force of the ground at the base of the ladder

f_s ... friction force at the base of the ladder

w_1 ... weight of the uniform ladder acting at its center

w_2 ... person's weight acting 1/3 of the ladder length from the base

ad a) <u>The first condition for equilibrium</u>

(I) $\sum F_x = f_s + (-n_1) = 0$

(II) $\sum F_y = n_2 + (-w_1) + (-w_2) = 0 \rightarrow n_2 = w_1 + w_2 =$

$= 180N + 800N = \underline{\underline{980N}}$

<u>The second condition for equilibrium</u>

Let's express the torques about the reference point $\underline{\underline{B}}$

$\rightarrow n_2, f_s$ have zero torques (zero lever arms about $\underline{\underline{B}}$)

(III) $\sum \tau_B = n_1 h + (-w_1 L_1) + (-w_2 L_2) = 0$

$\downarrow$

$$n_1 = \frac{w_1 L_1 + w_2 L_2}{h} = \frac{180 \cdot 1{,}5 + 800 \cdot 1}{4} N = \underline{\underline{268 N}}$$

• From (I): $f_s = n_1 = \underline{\underline{268\,N}}$

At the ladder base, the normal force is 980 N and the friction force is 268 N.

(ad b) (Maximal) static friction force at the base of the ladder is related to the normal force by $f_s = \mu_s^{min} \cdot n_2$

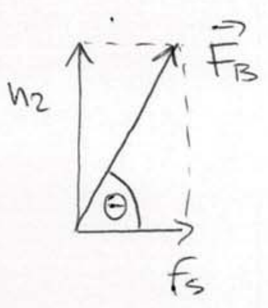
$$\downarrow$$

$$\mu_s^{min} = \frac{f_s}{n_2} = \frac{268\,N}{980\,N} = \underline{\underline{0{,}27}}$$

(ad c) At the base of the ladder the contact force has components $\underline{\underline{f_s}}$ and $\underline{\underline{n_2}}$:

$$\vec{F}_B = f_s\hat{\imath} + n_2\hat{\jmath} = (268N)\hat{\imath} + (980N)\hat{\jmath}$$

$$F_B = \sqrt{f_s^2 + n_2^2} = \sqrt{268^2 + 980^2}\;N = \underline{\underline{1016\,N}}$$

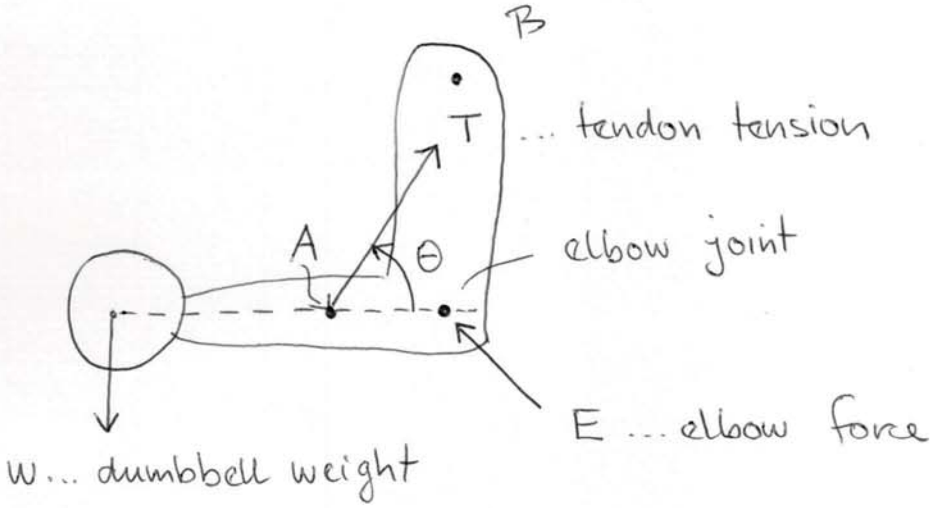


$$\tan\Theta = \frac{n_2}{f_s} \rightarrow \Theta = \arctan\frac{n_2}{f_s} =$$

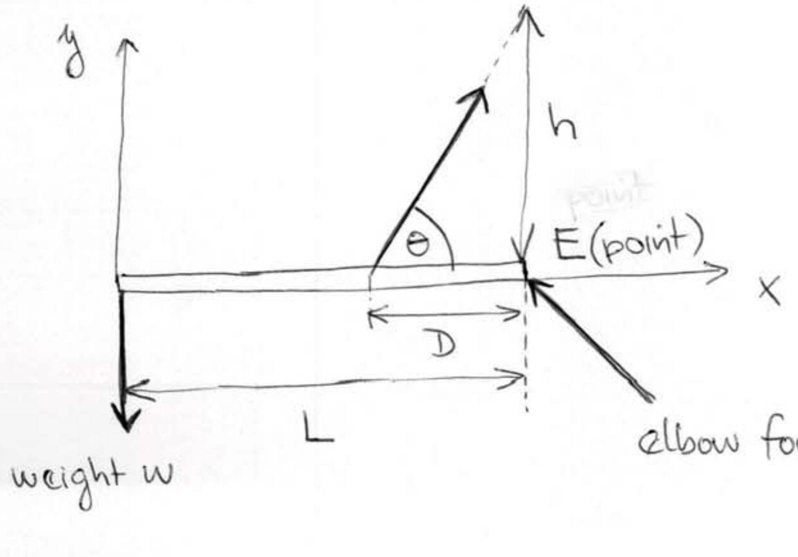
$$= \arctan\frac{980N}{268N} = \underline{\underline{74{,}7^\circ}}$$

The contact force at the ladder base is not directed along the ladder (otherwise, there would be a non-zero net torque about the top of the ladder.

Ex 11.4: When lifting a dumbbell, your forearm is in equilibrium under the action of the weight $\underline{\underline{w}}$ of the dumbbell, the tension $\underline{\underline{T}}$ in the tendon connected to the biceps muscle, and the force $\underline{\underline{E}}$ exerted by the upper arm at the elbow joint (see picture). Knowing the weight $\underline{\underline{w}}$ and the angle $\underline{\underline{\theta}}$ between the tension force and the horizontal, find the tendon tension and horizontal and vertical components of the elbow force $\underline{\underline{E}}$. Neglect the weight of the forearm itself.



Free-body diagram



L ... forearm length

h ... upper arm length

D ... distance between the tendon connecting point and forearm end

Tendon tension $\underline{\underline{T}}$: components $T_x = T \cdot \cos\theta$

$T_y = T \cdot \sin\theta$

Elbow force $\underline{\underline{E}}$: components E_x, E_y

The first condition for equilibrium

$$\sum F_x = T_x + (-E_x) = T \cdot \cos\theta - E_x = 0 \qquad \text{(I)}$$

$$\sum F_y = T_y + E_y + (-w) = T \cdot \sin\theta + E_y - w = 0 \qquad \text{(II)}$$

The second condition for equilibrium

- reference point for torque calculation at elbow joint
  → force E has zero torque (zero lever arm)

$$\sum \tau_E = L \cdot w + (-T_y D) = L \cdot w - T \cdot \sin\theta \, D = 0 \qquad \text{(III)}$$

- From (III): $\underline{\underline{T = \frac{L \cdot w}{D \cdot \sin\theta}}}$

- Inserting formula for $\underline{T}$ into (I) gives

$$E_x = T \cdot \cos\theta = \frac{L \cdot w}{D} \cdot \frac{\cos\theta}{\sin\theta} = \frac{L \cdot w}{D} \operatorname{cotg}\theta = \frac{L \cdot w}{\cancel{D}} \cdot \frac{\cancel{D}}{h} = \underline{\underline{\frac{L \cdot w}{h}}}$$

- Inserting formula for $\underline{T}$ into (II) gives

$$E_y = w - T \sin\theta = w - \frac{L w}{D \sin\theta} \cdot \sin\theta = w\left(1 - \frac{L}{D}\right) = -\underbrace{w}_{>0}\underbrace{\left(\frac{L-D}{D}\right)}_{>0}$$

↳ negative sign implies the real direction of E_y is downward

Section 11.4: Stress, strain, and elastic moduli

- Rigid body is an idealized model → real bodies stretch, squeeze, or twist when forces are applied to them

- Every kind of deformation can be described by:

(1) Stress ... characterizes the strength of the forces causing the deformation (force per unit area)

(2) Strain ... characterizes the amount of resulting deformation

- When stress and strain are sufficiently small, the two quantities are often directly proportional:

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{elastic modulus}}$$ Hooke's Law

elastic modulus ... constant equal to the stress required to produce unit strain

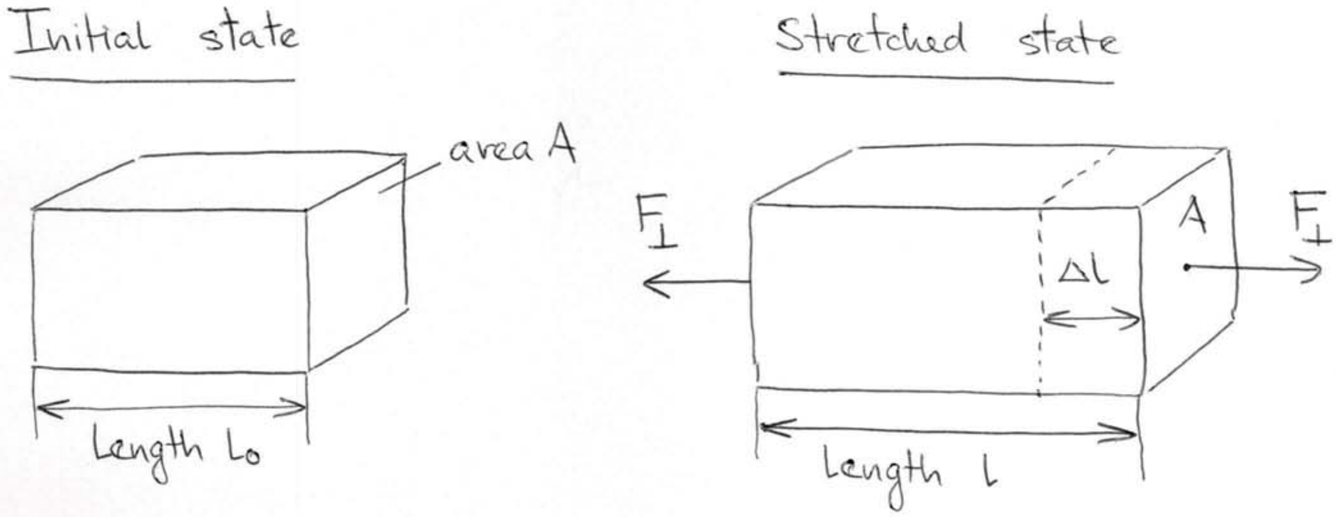
EG Ideal spring: $|F| = k \cdot y$

force (stress)　　extension / compression (strain)

<u>CAUTION</u>: Hooke's "law" is valid only for a limited range of deformations (see later)

Tensile and compressive stress and strain

- Consider stretching of a solid block subject to an external force



- initially, the block has a uniform cross-sectional area $\underline{\underline{A}}$ and length $\underline{\underline{L_0}}$
- when forces of equal magnitude $\underline{\underline{F_\perp}}$ and opposite directions are applied at the end faces of the block with area $\underline{\underline{A}}$, the net force acting on the block is zero $\rightarrow$ the block is in <u>tension</u> and has a new length $L > L_0$

$\downarrow$

Definition of <u>tensile stress</u>: $\boxed{\text{tensile stress} = \frac{F_\perp}{A}}$ $\rightarrow$ scalar

$F_\perp$ ... magnitude of the force <u>perpendicular</u> to the area $\underline{\underline{A}}$

SI units of stress: $1 \text{ pascal} = 1\,\text{Pa} = 1\,\frac{\text{N}}{\text{m}^2}$

$\hookrightarrow$ force per unit area

- Under tension, the block stretches:

$l = l_0 + \Delta l$

→ elongation Δl is distributed uniformly along the length of the block

↓

Definition of tensile strain : $\boxed{\text{tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}}$

↳ fractional change of the block length (dimensionless)

- For sufficiently small tensile stresses, stress and strain are directly proportional to each other - Hooke's law

↓

Definition of Young's modulus : $\boxed{Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F_\perp}{A} \frac{l_0}{\Delta l}}$

SI units of Young's modulus : $1\ Pa = 1 \frac{N}{m^2}$

Example : Young's modulus of some materials

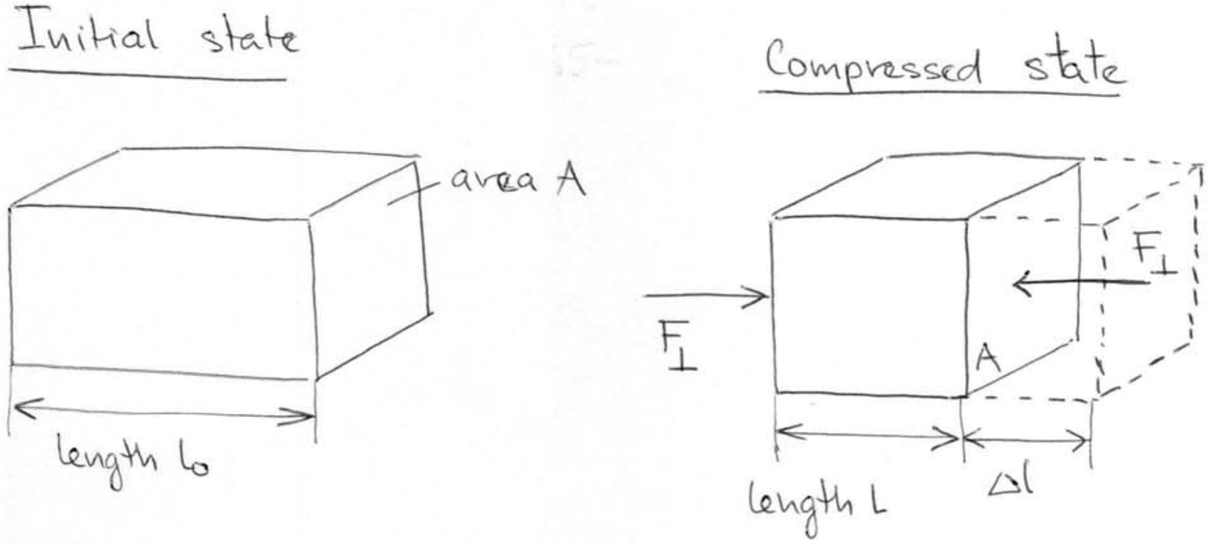
aluminum : $Y = 7{,}0 \times 10^{10}\ Pa = 70\ GPa$

glass : $Y = 6{,}0 \times 10^{10}\ Pa = 60\ GPa$

steel : $Y = 20 \times 10^{10}\ Pa = 200\ GPa$

→ the higher the value of Y, the higher the force needed to produce given stretching

- Consider now compression of a solid block subject to an external force



- when the applied external forces $F_\perp$ are directed inward (pushing forces), the net force on the block is again zero
  → the block is in compression

↓

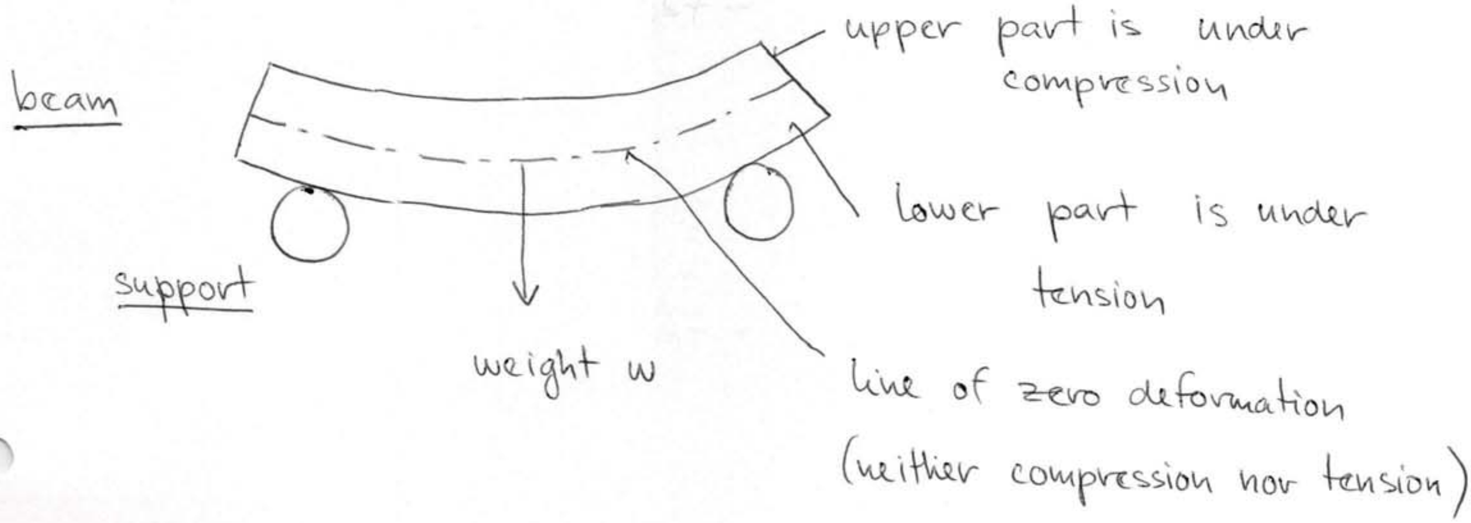
- Compressive stress and strain are defined in the same way as for the tensile stress and strain but Δl has the opposite direction

↓

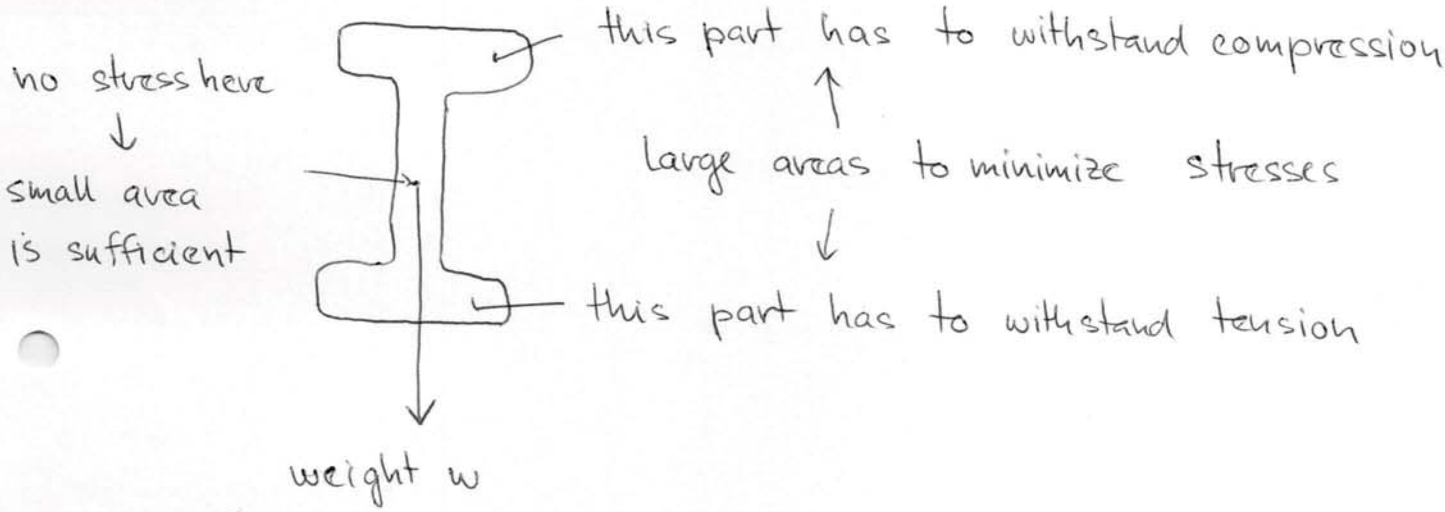
Hooke's law also applies to compression; in many cases, the value of Young's modulus is identical for tensile and compressive stresses

- Some materials (concrete, stone) can withstand compessive stress but break under a comparable tensile stress → this has to be considered in building design, e.g. by using arches to distribute forces

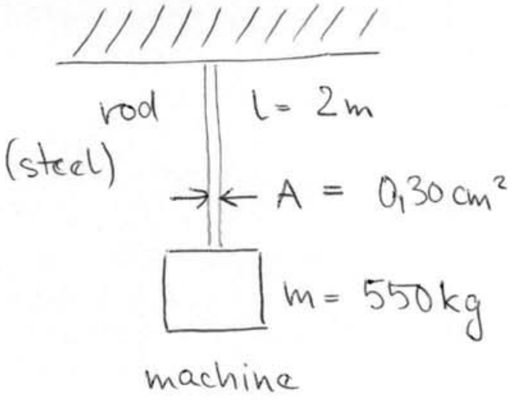
- Combined action of tensile and compressive stresses on the same body leads to <u>bending</u>



<u>beam cross-section</u> (I-beam)



<u>Ex 11.5</u>: A steel rod 2,0 m long has a cross-sectional area of $0{,}30\,\text{cm}^2$. The rod is connected with one end to the ceiling and a 550-kg machine is suspended from the rod's lower end. Determine the stress, the strain, and the elongation of the rod



$$\text{Stress} = \frac{F_\perp}{A} = \frac{m \cdot g}{A} = \frac{550 \cdot 9{,}8}{3 \times 10^{-5}}\ \text{Pa}$$

$$= \underline{\underline{1{,}8 \times 10^{8}\ \text{Pa}}}$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{\text{stress}}{Y} = \frac{1{,}8 \times 10^{8}}{20 \times 10^{10}} =$$

$$= \underline{\underline{9 \times 10^{-4}}}$$

Elongation: $\Delta l = \text{strain} \times l_0 = 9 \times 10^{-4} \cdot 2\,\text{m} = 0{,}0018\,\text{m} = \underline{\underline{1{,}8\,\text{mm}}}$

Bulk stress and strain

- Consider an object immersed in a fluid (liquid or gas) at rest
  ↳ fluid exerts a force $\underline{\underline{F_\perp}}$ on any part of the object's surface; the force on the object is perpendicular to the object's surface and pushes inward

⇓

Fluid pressure: $\boxed{p = \frac{F_\perp}{A}}$ → force of fluid per unit surface area of an immersed object

- fluid pressure increases with depth → it is the manifestation of fluid's own weight

↓

- for small enough objects, we can ignore these pressure variations and assume a constant pressure over the whole object's surface

SI units of pressure: 1 pascal $= 1\,\text{Pa} = 1\,\frac{\text{N}}{\text{m}^2}$

↳ the same as stress

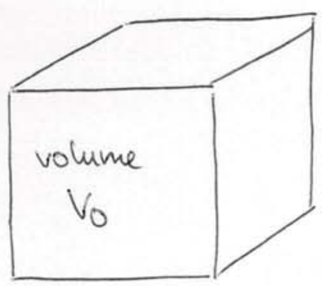
- Pressure is a <u>scalar</u> – it has no specific direction

- Pressure plays the role of stress in <u>volume deformations</u> (bulk)

→ pressure ⇒ volume (bulk) stress Δp

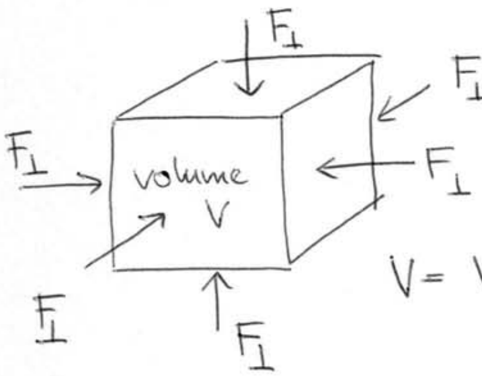
→ corresponding volume (bulk) strain $= \frac{\Delta V}{V_0} = \frac{V - V_0}{V_0}$

<u>Initial state</u>



pressure p_0

<u>Bulk stress</u>



$V = V_0 + \Delta V$

pressure $p = p_0 + \Delta p$

For $\Delta p > 0$, $\Delta V < 0$

- For sufficiently small bulk stresses and strains, Hooke's law applies

⇓

Definition of <u>bulk modulus</u>: $\boxed{B = \frac{\text{bulk stress}}{\text{bulk strain}} = -\frac{\Delta p}{\Delta V / V_0}}$

"minus" sign: increase of pressure Δp leads to decrease of volume ΔV, B is defined as a positive quantity

- Solids and liquids : $B \approx$ constant for small changes $\underline{\Delta p}$

$\times$

gases : $B \sim p_0$

- Reciprocal value of the bulk modulus is called the <u>compressibility</u>:

$$k = \frac{1}{B} = -\frac{\Delta V / V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p}$$

→ fractional decrease in volume $(-\Delta V/V_0)$ per unit increase in pressure (Δp)

→ SI units : Pa^{-1}

<u>Example</u> : Compressibility of some fluids

ethanol : $k = 110 \times 10^{-11}\ Pa^{-1}$

mercury : $k = 3{,}7 \times 10^{-11}\ Pa^{-1}$

water : $k = 45{,}8 \times 10^{-11}\ Pa^{-1}$

<u>Ex 11.6</u>: A hydraulic press contains 250 liters of oil. Find the decrease of oil volume when the pressure of oil increases by $\Delta p = 1{,}6 \times 10^{7}$ Pa. The bulk modulus of the oil is $B = 5{,}0 \times 10^{9}$ Pa.

From the definition of B:

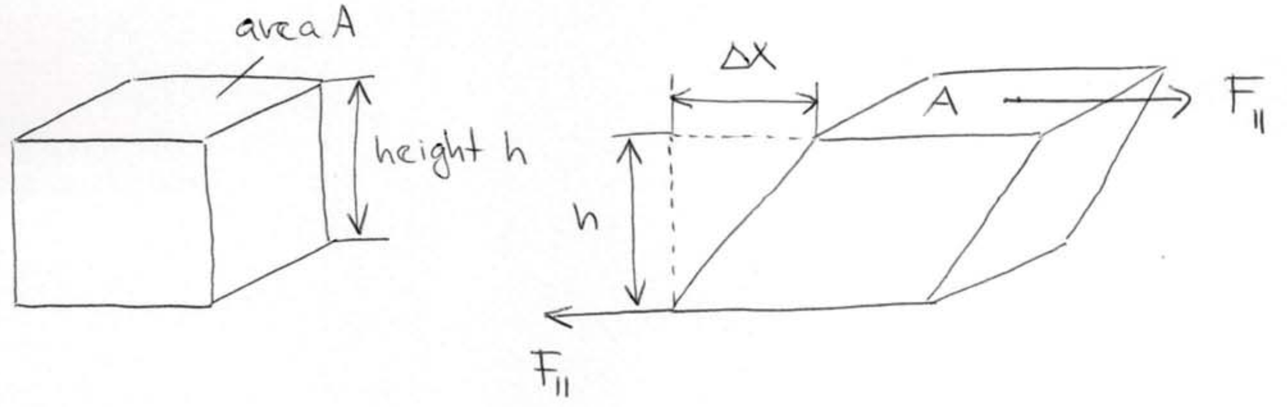
$$B = -\frac{\Delta p}{\Delta V/V_0} \Rightarrow \Delta V = -\frac{\Delta p\, V_0}{B} = -\frac{1{,}6 \times 10^7 \cdot 0{,}250}{5{,}0 \times 10^9}\, m^3$$

$$= -8 \times 10^{-4}\, m^3 = \underline{\underline{-0{,}80\, L}}$$

Shear stress and strain

- Shear occurs when forces of equal magnitude and opposite directions act tangent to the surfaces of opposite ends of an object

Initial state | Sheared state



Definition of shear stress: $\boxed{\text{shear stress} = \frac{F_{\parallel}}{A}}$ force per unit area

$F_{\parallel}$ ... magnitude of the force parallel to the area $\underline{\underline{A}}$

Definition of shear strain: $\boxed{\text{shear strain} = \frac{\Delta x}{h}}$ Typically $\Delta x \ll h$

↳ shear leads to changes of angles within deformed bodies

- For small forces / deformations, Hooke's law applies:

$\downarrow$

Definition of <u>shear modulus</u>: $\boxed{S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F_{\parallel}}{A}\frac{h}{\Delta x}}$

- Typically, shear modulus $\underline{\underline{S}}$ is about one-half to one-third as large as Young's modulus $\underline{\underline{Y}}$

<u>Example</u>:

	steel	glass	aluminum
Y [Pa]	20×10^{10}	6×10^{10}	7×10^{10}
S [Pa]	$7{,}5 \times 10^{10}$	$2{,}5 \times 10^{10}$	$2{,}5 \times 10^{10}$

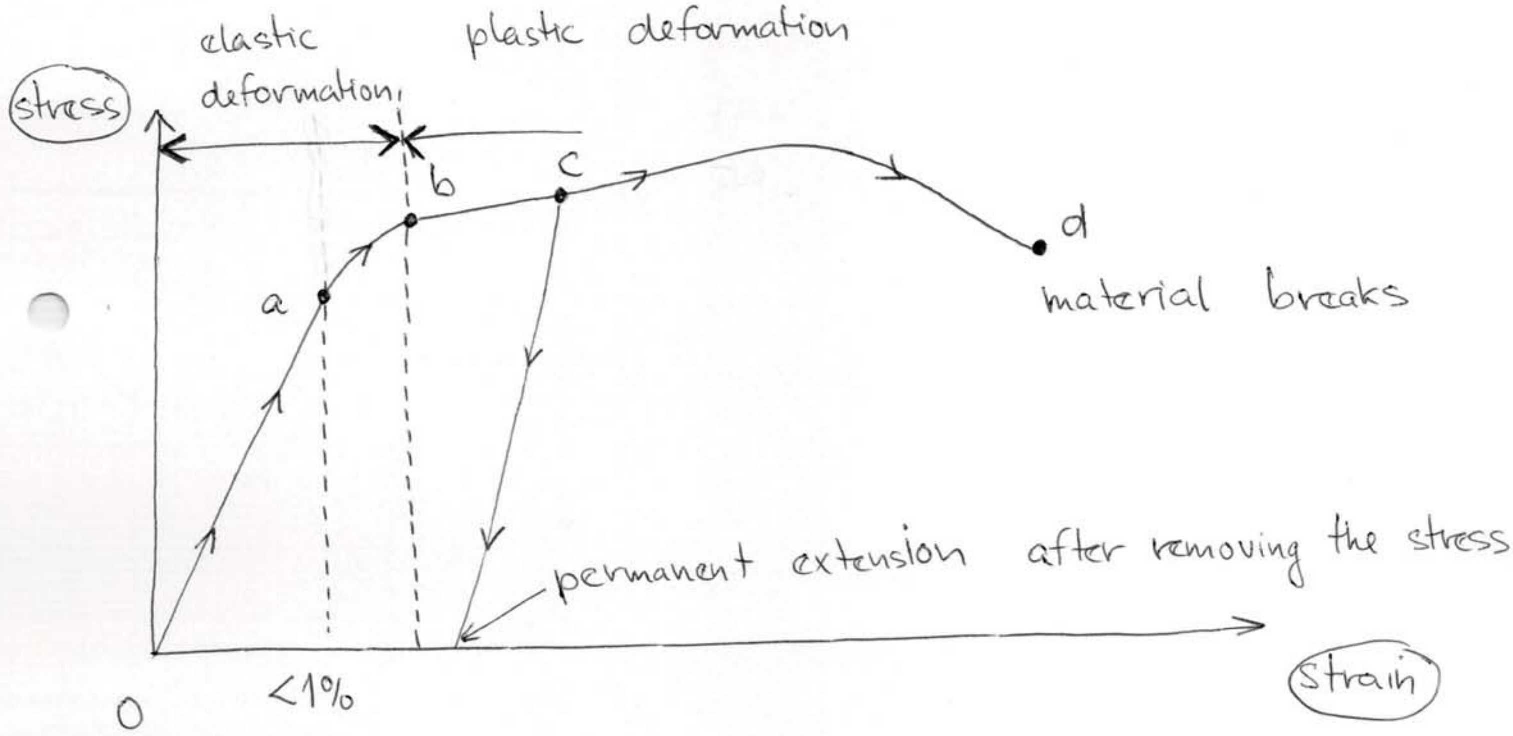
- Shear modulus concept only applies to solids that can return to their original shape when the shear forces are removed

$\times$

Gasses and liquids do not have definite shapes and respond to shear forces by dissipation (fluid friction)

Section 11.5: Elasticity and plasticity

- Hooke's law of proportionality between stress and strain applies only to small forces and deformations



$0 \to a$: Hooke's law is valid - stress/strain curve is a straight line

$a \to b$: Hooke's law is not obeyed but the deformation is still elastic $\Rightarrow$ after removing the force, the object returns to its original shape $\Rightarrow$ deformation is reversible

b: yield point $\to$ strains beyond this point are irreversible
$\Rightarrow$ permanent change of object's shape
$\to$ maximal stress at yield point = elastic limit

b → d: plastic flow (deformation) – material keeps elongating with a minimal change of applied stress, elongation is permanent and persists after removing the deforming forces

d: fracture point → material breaks

→ maximal stress a given material can withstand : ultimate / tensile strength