Introduction to Topology Midterm Exam, March 29th, 2016

INSTRUCTIONS: Do all of the problems in PART I. Do only two problems from PART II. Questions in PART I all worth 12 points. Questions in PART II all worth 20 points.

PART I

1. Write the definitions of the following terms:

   (a) $\lim_{n\to\infty} a_n = a$:

   > **Solution:** For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $|a_n - a| < \epsilon$.

   (b) $A \subseteq \mathbb{R}$ is a connected set:

   > **Solution:** For every $a, b \in A$, and for every $x \in \mathbb{R}$ if $a \leq x \leq b$ then $x \in A$. Or, equivalently, for every $a, b \in A$ we also have $[a, b] \subseteq A$.

   (c) $A \subseteq \mathbb{R}$ is an open set:

   > **Solution:** For every $a \in A$ there is a positive number $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subseteq A$.

   (d) $f : \mathbb{R} \to \mathbb{R}$ is continuous:

   > **Solution:** There are 3 equivalent definitions:
   >
   > 1. For every sequence (x_n) in $\mathbb{R}$ if there is a real number such that $x = \lim_{n\to\infty} x_n$ then we also have $f(x) = \lim_{n\to\infty} f(x_n)$.
   > 2. For every $x \in \mathbb{R}$ and positive real number $\epsilon > 0$ there is a positive real number $\delta > 0$ such that for every $y \in \mathbb{R}$ if $|x - y| < \delta$ then $f|(x) - f(y)| < \epsilon$.
   > 3. For every $O \subseteq \mathbb{R}$, if O is open then so is $f^{-1}(O)$.

2. Assume $f : X \to Y$ is a function. Recall that for every $U \subseteq Y$ we define

$$f^{-1}(U) = \{x \in X |\ f(x) \in U\}$$

   (a) Show that for every $U \subseteq Y$ we have

$$f^{-1}(U)^c = f^{-1}(U^c)$$

   where U^c is the complement $U^c := Y \setminus U$.

   > **Solution:** For every $x \in \mathbb{R}$, we have $x \in f^{-1}(U^c)$ if and only if (by definition) $f(x) \in U^c$, i.e. $f(x) \notin U$. This is equivalent to saying $x \notin f^{-1}(U)$, which in turn is equivalent to $x \in f^{-1}(U)^c$.

   (b) Show that for every family of sets $\{U_i\}_{i\in I}$ in Y we have

$$\bigcup_{i\in I} f^{-1}(U_i) = f^{-1}\left(\bigcup_{i\in I} U_i\right)$$

Solution: An element $x \in \mathbb{R}$ belongs to $\bigcup_{i\in I} f^{-1}(U_i)$ when there is an index $j \in I$ such that $x \in f^{-1}(U_j)$. However, this means that $f(x) \in U_j$ for the same index $j \in I$. Hence $f(x) \in \bigcup_{i\in I} U_i$, i.e. $x \in f^{-1}\left(\bigcup_{i\in I} U_i\right)$. This proves $\bigcup_{i\in I} f^{-1}(U_i) \subseteq f^{-1}\left(\bigcup_{i\in I} U_i\right)$. For the reverse inclusion, take $x \in f^{-1}\left(\bigcup_{i\in I} U_i\right)$. Then $f(x) \in \bigcup_{i\in I} U_i$ which means there is an index $j \in I$ with $f(x) \in U_j$. Thus $x \in f^{-1}(U_j)$ for the same index, i.e. $x \in \bigcup_{i\in I} f^{-1}(U_i)$ proving the reverse inclusion $f^{-1}\left(\bigcup_{i\in I} U_i\right) \subseteq \bigcup_{i\in I} f^{-1}(U_i)$.

(c) Using (a) and (b) show that

$$\bigcap_{i\in I} f^{-1}(U_i) = f^{-1}\left(\bigcap_{i\in I} U_i\right)$$

Solution:

$$\bigcap_{i\in I} f^{-1}(U_i) = \left(\bigcup_{i\in I} f^{-1}(U_i)^c\right)^c = \left(\bigcup_{i\in I} f^{-1}(U_i^c)\right)^c = \left(f^{-1}\left(\bigcup_{i\in I} U_i^c\right)\right)^c$$
$$= f^{-1}\left(\left(\bigcup_{i\in I} U_i^c\right)^c\right) = f^{-1}\left(\bigcap_{i\in I} U_i\right)$$

3. Show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Solution: Assume $\epsilon > 0$ is given. We would like to prove that there is an index $N \in \mathbb{N}$ such that for every index with $n \geq N$ we have $|\frac{1}{\sqrt{n}}| < \epsilon$. Since $n > 0$, it is enough to prove $\frac{1}{\sqrt{n}} < \epsilon$. So, we need:

$$\frac{1}{\sqrt{n}} < \epsilon \iff \frac{1}{\epsilon} < \sqrt{n} \iff \frac{1}{\epsilon^2} < n$$

So, if $N > \lceil \frac{1}{\epsilon^2} \rceil$ then for every $n \geq N$ we also have $n > \lceil \frac{1}{\epsilon^2} \rceil$ which gives what we are looking for.

4. Recall that given $A \subseteq \mathbb{R}$, we define $d(x, A) = \inf_{a\in A} |x - a|$. Calculate the function $d(x, A)$ for the following subsets:

(a) $A = (-\infty, -1] \cup [1, \infty)$

Solution:

$$d(x, A) = \begin{cases} 0 & \text{if } x \leq -1 \text{ or } x \geq 1 \\ x - 1 & \text{if } -1 \leq x \leq 0 \\ 1 - x & \text{if } 0 \leq x \leq 1 \end{cases}$$

(b) $A = \bigcup_{n\in\mathbb{Z}} [2n, 2n+1]$

Solution:

$$d(x, A) = \begin{cases} 0 & \text{if } 2n \leq x \leq 2n+1 \text{ for some } n \in \mathbb{Z} \\ x - 2n + 1 & \text{if } 2n-1 \leq x \leq 2n - \frac{1}{2} \text{ for some } n \in \mathbb{Z} \\ 2n + 1 - x & \text{if } 2n + \frac{1}{2} \leq x \leq 2n+1 \text{ for some } n \in \mathbb{Z} \end{cases}$$

[**Hint**: Sketch the graphs of these functions first, then write an expression.]

5. Show only one of the following statements:

   (a) Let $\{A_i\}_{i\in I}$ be a family of closed sets in $\mathbb{R}$. Show that $\bigcap_{i\in I} A_i$ is also closed.

   Solution: Assume $\{A_i\}_{i\in I}$ is a family of closed sets in $\mathbb{R}$, and let (x_n) be a sequence in $\bigcap_{i\in I} A_i$ such that there is a real number $x \in \mathbb{R}$ with $x = \lim_{n\to\infty} x_n$. Then for every $n \in \mathbb{N}$ and for every $i \in I$ we have $x_n \in A_i$. Since each A_i is closed, we must have $x \in A_i$ for every $i \in I$. In other words, $\bigcap_{i\in I} A_i$ must be closed.

   (b) Let $\{U_i\}_{i\in I}$ be a family of open sets in $\mathbb{R}$. Show that $\bigcup_{i\in I} U_i$ is also open.

   Solution: Assume $\{U_i\}_{i\in I}$ is a family of open sets in $\mathbb{R}$ and let $x \in \bigcup_{i\in I} U_i$. Then there is an index $j \in I$ such that $x \in U_j$. However, U_j is open. This means there a positive number $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U_j$. Since $U_j \subseteq \bigcup_{i\in I} U_i$ we see that $(x-\epsilon, x+\epsilon) \subseteq \bigcup_{i\in I} U_i$. This proves $\bigcup_{i\in I} U_i$ is open.

PART II

6. Consider the set of all sequences of real numbers

$$\mathcal{S} = \{(a_n)_{n\in\mathbb{N}} : \ a_n \in \mathbb{R}\}$$

Define a relation $A\colon \mathcal{S} \to \mathcal{S}$ such that

$$A = \left\{((a_n), (b_n)) \in \mathcal{S} \times \mathcal{S} : \ \lim_{n\to\infty} |a_n - b_n| = 0\right\}$$

In other words, two sequences are related when their difference converges to 0. Verify that this relation is an equivalence relation.

Solution:

1. A is reflexive because for every $(x_n) \in \mathcal{S}$ we have $0 = \lim_{n\to\infty} |x_n - x_n|$.
2. A is symmetric because if $((x_n), (y_n)) \in A$ then $0 = \lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} |y_n - x_n|$ which implies $((y_n), (x_n)) \in A$.
3. A is transitive: Notice that $\lim_{n\to\infty} |x_n - y_n| = 0$ means $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. So, if we have $((x_n), (y_n)) \in A$ and $((y_n), (z_n)) \in A$, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \quad \text{and} \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$$

   Then we can conclude that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n$, i.e. $((x_n), (z_n)) \in A$.

7. Assume $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, and that there is a number $a \in \mathbb{R}$ such that $f(a) > 0$. Show that there is a $\delta > 0$ such that for every $x \in (a-\delta, a+\delta)$ we also have $f(x) > 0$.

> **Solution:** The set $(0,\infty)$ is an open subset and $a \in f^{-1}(0,\infty)$ since $f(a) > 0$. The continuity of f says that $f^{-1}(0,\infty)$ is also an open set. Then there is a $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq f^{-1}(0,\infty)$. Then for every $x \in (a-\delta, a+\delta)$ we have $f(x) > 0$.

8. Assume (a_n) is a sequence of real numbers. One defines *the sequence of partial sums* as

$$S_N := \sum_{n=0}^{N} a_n$$

and we say that *the series* $\sum_{n=0}^{\infty} a_n$ *is convergent* when the sequence $(S_N)_{N\in\mathbb{N}}$ is convergent. Show that the series $\sum_{n=0}^{\infty} a$ is divergent for any $a > 0$. [**Hint**: Write the sequence of partial sums first, and then show that that series diverges to ∞.]

> **Solution:** First, recall that we say that a sequence (a_n) diverges to ∞ when for every $\epsilon > 0$ there is an index $N > 0$ such such that for every $n \geq N$ we have $a_n > \epsilon$. The sequence of partial sums for the series $\sum_{n=0}^{\infty} a$ is
>
> $$S_N = \sum_{n=0}^{N} a = (N+1)a$$
>
> Then for every $\epsilon > 0$ there is an index $N > 0$ such that $(n+1)a > \epsilon$ for every $n \geq N$ if we choose $N = \left\lceil \frac{\epsilon}{a} - 1 \right\rceil$. This is because if $N = \left\lceil \frac{\epsilon}{a} - 1 \right\rceil$ then
>
> $$n \geq N = \left\lceil \frac{\epsilon}{a} - 1 \right\rceil \geq \frac{\epsilon}{a} - 1 \Longrightarrow (n+1)a > \epsilon$$
>
> as we wanted to show.

9. Assume $x \in \mathbb{R}$ is an arbitrary real number. Let $\mathcal{O}_x$ the set of all open subsets containing this element x. Show that

$$\bigcap_{U\in\mathcal{O}_x} U = \{x\}$$

[**Hint**: Use "proof by contradiction." What would happen if there is a real number $y \neq x$ inside $\bigcap_{U\in\mathcal{O}_x} U$? Construct an open set which contains x but not y. How would this contradict your assumption?]

> **Solution:** First, notice that for every $U \in \mathcal{O}_x$ we have $x \in U$. So, $\{x\} \subseteq U$ for every $U \in \mathcal{O}_x$ which means
>
> $$\{x\} \subseteq \bigcap_{U\in\mathcal{O}_x} U$$
>
> Assume now that there is another element $y \in \bigcap_{U\in\mathcal{O}_x} U$ such that $y \neq x$. If we define $\delta = \frac{|x-y|}{2}$ we see that $\delta > 0$. Then $(x-\delta, x+\delta) \in \mathcal{O}_x$ is an open set containing x but not y. Now notice that $y \in \bigcap_{U\in\mathcal{O}_x} U \subseteq (x-\delta, x+\delta)$ because the intersection $\bigcap_{U\in\mathcal{O}_x} U$ is contained in every $U \in \mathcal{O}_x$. We got a contradition: on one hand we started with $x \neq y \in \bigcap_{U\in\mathcal{O}_x} U$ and now we concluded $y \in (x-\delta, x+\delta)$ which does not contain y. So, our assumption that "there is a $y \in \bigcap_{U\in\mathcal{O}_x}$ with $y \neq x$" was false. Therefore the set $\bigcap_{U\in\mathcal{O}_x}$ contains only x.