Operations Research
Integer Programming

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Integer Programming

- When formulating LP’s we often found that, strictly, certain variables should have been regarded as taking integer values but, for the sake of convenience, we let them take fractional values reasoning that the variables were likely to be so large that any fractional part could be neglected.
- While this is acceptable in some situations, in many cases it is not, and in such cases we must find a numeric solution in which the variables take integer values.
- Problems in which this is the case are called integer programs (IP's) and the subject of solving such programs is called integer programming (also referred to by the initials IP).

- An IP in which all variables are required to be integers is called a pure IP problem.
- If some variables are restricted to be integer and some are not then the problem is a mixed IP problem
- The case where the integer variables are restricted to be 0 or 1 comes up surprising often. Such problems are called pure (mixed) 0-1 programming problems or pure (mixed) binary IP problems.

Formulation

- The same as LP formulation except integer or binary variables are indicated seperately

- Eg. (Pure IP)

$$
\begin{aligned}
\max z &= x_1 + x_2 + x_3 \\
\text{s.t.} \quad & x_1 + 6x_2 + x_3 \leq 8 \\
& x_1 + 2x_2 + 1{,}5x_3 \leq 2 \\
& 2x_1 + x_2 + 5x_3 \leq 8 \\
& x_1, x_2, x_3 \ \text{integer}
\end{aligned}
$$

Formulation

- Eg. (Pure 0-1)

$$\max z = 2x_1 + 3x_2 + 4x_3 + 7x_4 + 2x_5$$

$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 \leq 8$$

$$x_1, x_2, x_3, x_4, x_5 = 0 \text{ or } 1$$

Formulation

- Eg. Mixed Integer Program

$$\max z = x_1 - 2x_2 + x_3 + 2x_4$$

$$\text{s.t.} \quad x_1 + 6x_2 + x_3 - 2x_4 \leq 8$$

$$x_1 + 2x_2 + 1{,}5x_3 + x_4 \leq 2$$

$$2x_1 + x_2 + 5x_3 + 2x_4 \geq 8$$

$$x_1 \geq 0; \quad x_2, x_3 \text{ integer}; \quad x_4 \text{ 0 or 1}$$

LP relaxation

- For any IP we can generate an LP by taking the same objective function and same constraints but with the requirement that variables are integer replaced by appropriate continuous constraints:
  - "$x_i \geq 0$ and integer" can be replaced by $x_i \geq 0$
  - "$x_i = 0$ or 1" can be replaced by $x_i \geq 0$ and $x_i \leq 1$
- The LP obtained by omitting all integer or 0-1 constraints on variables is called LP Relaxation of the IP (LR).

LP relaxation

Mixed Integer Programming

$\max z = x_1 + x_2 + x_3$

s.t. $x_1 + 6x_2 + x_3 \leq 8$

$x_1 + 2x_2 + 1,5x_3 \leq 2$

$2x_1 + x_2 + 5x_3 \leq 8$

$x_1 \geq 0$, x_2 integer, x_3 binary

LP relaxation of the model

$\max z = x_1 + x_2 + x_3$

s.t. $x_1 + 6x_2 + x_3 \leq 8$

$x_1 + 2x_2 + 1,5x_3 \leq 2$

$2x_1 + x_2 + 5x_3 \leq 8$

$x_3 \leq 1$

$x_1, x_2, x_3 \geq 0$

Optimum solution of LP Relaxation model

is always better than or as good as

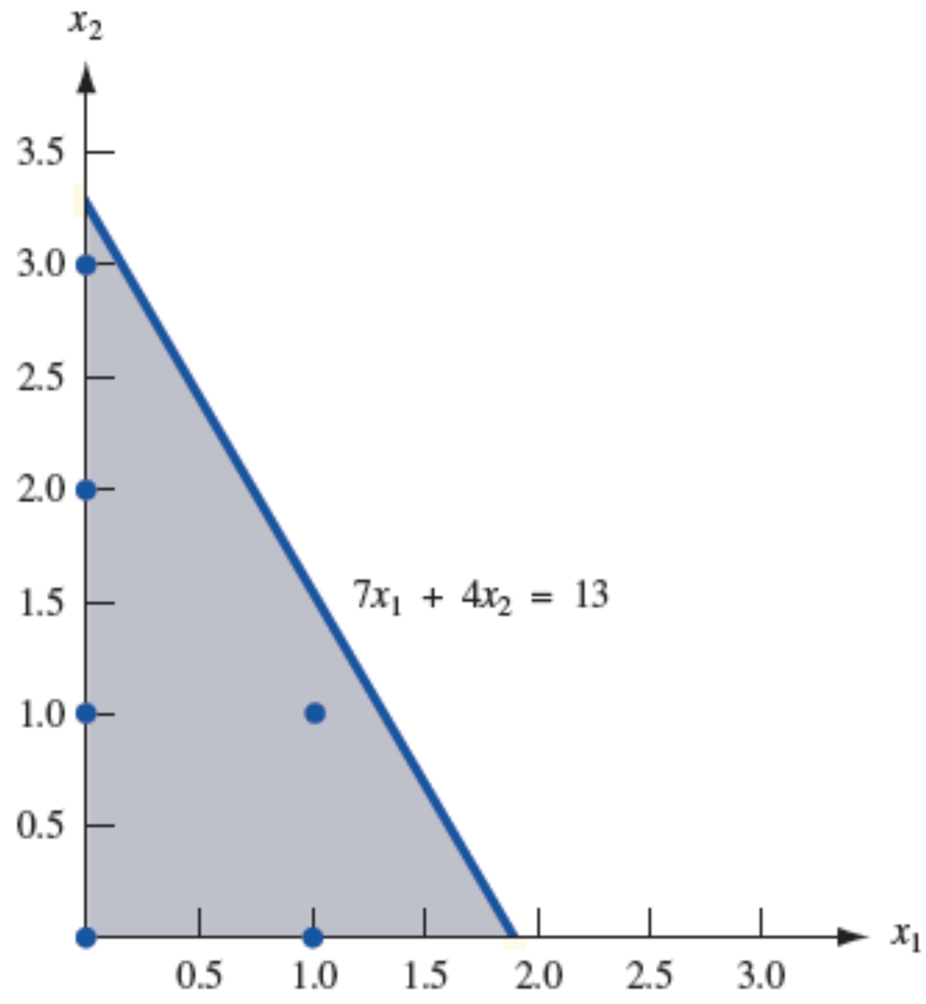
Optimum solution of IP model

Example

$\max z = 21x_1 + 11x_2$

s.t. $7x_1 + 4x_2 \leq 13$

$x_1,\ x_2$ integer



Formulating IP

- Practical problems can be formulated as IPs.
- For instance
  - budgeting problems,
  - knapsack problems,
  - fixed charge production and location problems,
  - set covering problems,
  - etc.

Budgeting problems

- <u>Capital Budgeting Example</u>
- Stock is considering four investments
- Each investment yields a determined NPV ($8,000, $11,000, $6,000, $4,000)
- Each investment requires at certain cash flow at the present time ($5,000, $7,000, $4,000, $3,000)
- Currently Stock has $14,000 available for investment.
- Formulate an IP whose solution will tell Stock how to maximize the NPV obtained from the four investments.

Capital Budgeting Example Answer

- max $z = 8x_1 + 11x_2 + 6x_3 + 4x_4$
- s.t. $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$
- $x_j = 0$ or 1 ($j = 1,2,3,4$)

- This is a knapsack problem!
  - The traditional story is that: There is a knapsack. There are a number of items, each with a size and a value. The objective is to maximize the total value of the items in the knapsack.
  - Knapsack problems are nice because they are (usually) easy to solve.
- Max $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
- S.t. $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$
- $x_i = 0$ or 1 ($i = 1, 2, \ldots, n$)

Capital Budgeting - Multiperiod

- There are four possible projects, which each run for three years and have the following characteristics:

		Capital requirements		
Project	Return	Year1	Year2	Year3
1	0.2	0.5	0.3	0.2
2	0.3	1	0.5	0.2
3	0.5	1.5	1.5	0.3
4	0.1	0.1	0.4	0.1
Available capital		3.1	2.5	0.4

- Which projects would you choose in order to maximize the total return?

Capital Budgeting Extension

- Stock can only make two investments
- If investment 2 is made, investment 4 must also be made
- If investment 1 is made, investment 3 cannot be made
- Either investment 1 or investment 2 must be done

Fixed Charge Problems

- There is a cost associated with performing an activity at a nonzero level that does not depend on the level of the activity.
- An important trick can be used to formulate many production and location problems involving the idea of a fixed charge as IP.

Gandhi

- Gandhi Co makes shirts, shorts, and pants using the limited labor and cloth described below.
- In addition, the machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, $200 per week; shorts machinery, $150 per week; pants machinery, $100 per week.
- Each week, 150 hours of labor and 160 sq yd of cloth are available.
- Formulate an IP whose solution will maximize Gandhi's weekly profits.

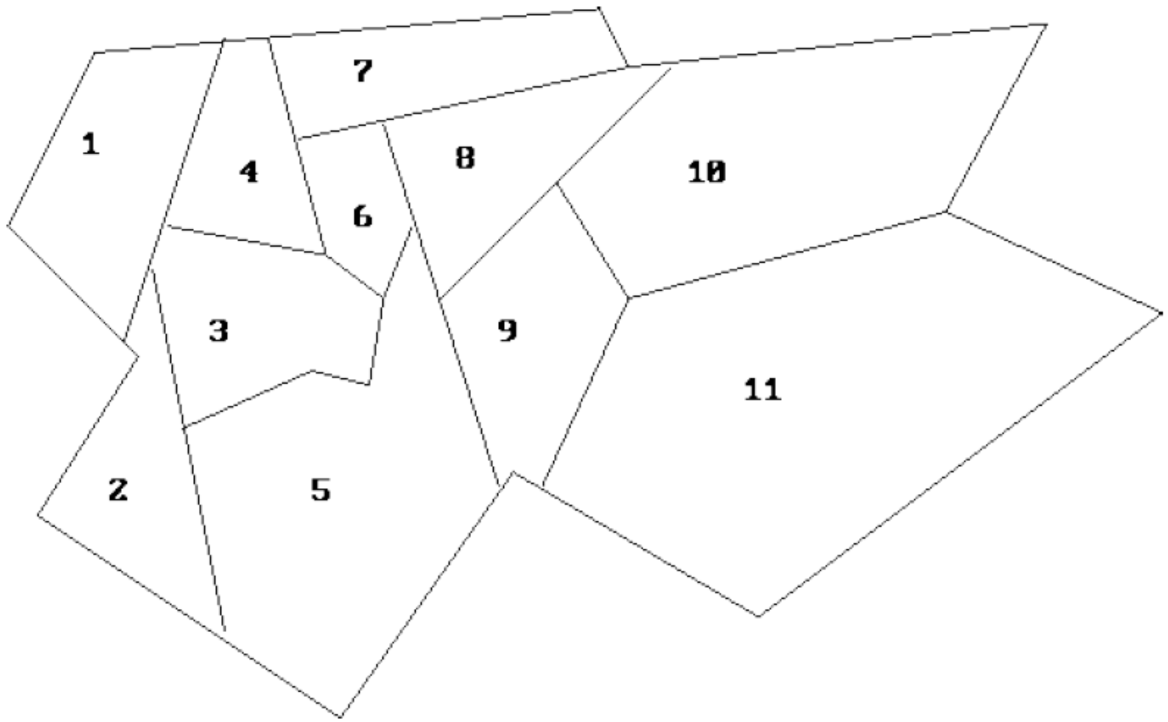
Product	Labor (hrs/wk)	Cloth (m2/wk)	Sale Price	Variable unit cost
Shirts	3	4	12	6
Shorts	2	3	8	4
Pants	6	4	15	8

Set Covering Problems

- Each member of a given set (call it set 1) must be "covered" by an acceptable member of some set (call it set 2).
- The objective of a set-covering problem is to minimize the number of elements in set 2 that are required to cover all the elements in set 1.

Fire Station example

- A county is reviewing the location of its fire stations.
- The county is made up of a number of cities:
- A fire station can be placed in any city.
- It is able to handle the fires for both its city and any adjacent city (any city with a non-zero border with its home city).
- How many fire stations should be built and where?



Either-Or Constraints

- Given two constraints ;
- $f(x_1, x_1, \dots, x_n) \leq 0$ (1)
- $g(x_1, x_1, \dots, x_n) \leq 0$ (2)

- ensure that at least one is satisfied (1 or 2) by adding either-or-constraints:
- $f(x_1, x_1, \dots, x_n) \leq My$
- $g(x_1, x_1, \dots, x_n) \leq M(1 - y)$
- $y = 0 \; or \; 1$

Either-Or Constraints Example

- Dorian produces 3 types of cars. Required steel and required hours of labor for each type is given in the table.
- At present, 6,000 tons of steel and 60,000 hours of labor are available.
- For an economically feasible production, at least 1,000 cars of each type of a car must be produced.
  - Constraint: $x \leq 0$ or $x \geq 1000$
  - Sign restriction: $x \geq 0$ and Integer

	Compact car	Hatchback	Stationwagon
Required steel	1,5 tons	3 tons	5 tons
Required labor	30 hours	25 hours	40 hours
Unit profit ($)	2.000	3.000	4.000

If-Then Constraints

- Suppose we want to ensure that
  - a constraint $f(x_1, x_1, \dots, x_n) > 0$ implies
  - the constraint $g(x_1, x_1, \dots, x_n) \geq 0$
- Then we include the following constraints in the formulation:
  - $-g(x_1, x_1, \dots, x_n) \leq My$ (1)
  - $f(x_1, x_1, \dots, x_n) \leq M(1 - y)$ (2)
  - $y = 0\ or\ 1$

- Here y is a 0-1 variable, and M is a large positive number.
- If $f > 0$, then (2) can be satisfied only if $y = 0$. (1) implies $-g \leq 0$ or $g \geq 0$, which is the desired result.
- If $f(x) > 0$ is not satisfied $g(x) \geq 0$ can be satisfied or not.

If-Then Constraints Example

- In the Dorian Example;
- Suppose if Stationwagon is produced then at most 2000 units of other types can be produced.

Example 1

- A Sunco oil delivery truck contains five compartments, holding up to 2,700, 2,800, 1,100, 1,800, and 3,400 gallons of fuel, respectively.
- The company must deliver three types of fuel (super, regular, and unleaded) to a customer.
- The demands, penalty per gallon short, and the maximum allowed shortage are given in the Table.
- Each compartment of the truck can carry only one type of gasoline.
- Formulate an IP whose solution will tell Sunco how to load the truck in a way that minimizes shortage costs.

Type of Gasoline	Demand	Cost per Gallon Short - $	Maximum allowed Shortage
Super	2900	10	500
Regular	4000	8	500
Unleaded	4900	6	500

Example 2

- The Research and Development Division of the Progressive Company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, an operations research study has been requested to find the most profitable product mix.
- A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. Management's objective is to find the product mix that maximizes the total profit (total net revenue minus start-up costs).
- Let the continuous decision variables x_1, x_2, x_3, and x_4 be the production levels of products 1, 2, 3, and 4, respectively. Management has imposed the following policy constraints on these variables:
  - 1. No more than two of the products can be produced.
  - 2. Either product 3 or 4 can be produced only if either product 1 or 2 is produced.
  - 3. Either $5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6{,}000$
  - or $4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6{,}000$.
- Formulate a mixed BIP model for this problem.

	Product			
	1	2	3	4
Start-up cost	$50,000	$40,000	$70,000	$60,000
Marginal revenue	$ 70	$60	$90	$80

Example 3

- The ATK Toys Company has developed two new toys for possible inclusion in its product line for the upcoming Christmas season. Setting up the production facilities to begin production would cost $50,000 for toy 1 and $80,000 for toy 2. Once these costs are covered, the toys would generate a unit profit of $10 for toy 1 and $15 for toy 2.
- The company has two factories that are capable of producing these toys. However, to avoid doubling the start-up costs, just one factory would be used, where the choice would be based on maximizing profit. For administrative reasons, the same factory would be used for both new toys if both are produced.
- Toy 1 can be produced at the rate of 50 per hour in factory 1 and 40 per hour in factory 2. Toy 2 can be produced at the rate of 40 per hour in factory 1 and 25 per hour in factory 2.
- Factories 1 and 2, respectively, have 500 hours and 700 hours of production time available before Christmas that could be used to produce these toys.
- It is not known whether these two toys would be continued after Christmas. Therefore, the problem is to determine how many units (if any) of each new toy should be produced before Christmas to maximize the total profit.
- Formulate an MIP model for this problem.

Example 4

- Glueco produces three types of glue on two different production lines. Each line can be utilized by up to seven workers at a time.
- Workers are paid $500 per week on production line 1, and $900 per week on production line 2.
- A week of production costs $1,000 to set up production line 1 and $2,000 to set up production line 2.
- During a week on a production line, each worker produces the number of units of glue shown in the following Table.
- Each week, at least 120 units of glue 1, at least 150 units of glue 2, and at least 200 units of glue 3 must be produced.
- Formulate an IP to minimize the total cost of meeting weekly demands.

Production line	Glue 1	Glue 2	Glue 3
1	20	30	40
2	30	35	45